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Asymptotic of the geometric mean error in the quantization of recurrent self-similar measures

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Assuming a separation property, we determine the quantization dimension $D(\mu)$ of a recurrent self-similar measure μ with respect to the geometric mean error, and prove that $D(\mu)$ coincides with the Hausdorff dimension $\dim_{\mathrm{H}}^{*}(\mu)$ of μ.

1. Introduction

The aim of quantization for probability measures is to approximate a given probability measure with discrete probability measure with finite support. The quantization problem has its origins in information theory and engineering technology (see [4,10,13,18]). This problem has been studied intensively in recent years both for absolute continuous probability measures and for singular measures supported on fractal sets. Much of the previous work in quantization is based on L_r-metrics ($0 < r < +\infty$) as a measure of quantization error. The mathematical foundation of the quantization problem for these metrics is treated in Graf–Luschgy's book (see [11]). One of the mathematical goals of the quantization problem based on L_r-metrics is to study the asymptotic property for the error when approximating a given probability measure of finite support in the sense of L_r-metrics. For recent investigations of special aspects of L_r-quantization one can see [15,16]. Since the L_r-metric converges to the geometric mean as $r \to 0$ it is natural to look at the quantization problem with respect to the geometric mean error.

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Given a Borel probability measure μ on $\mathbb{R}^d$, the nth quantization error for μ with respect to the geometric mean error is given by

$$e_n(\mu) := \inf\Big\{\exp\int \log d(x,\alpha)d\mu(x) : \alpha \subset \mathbb{R}^d,\ 1 \leq \text{card}(\alpha) \leq n\Big\}, \tag{1}$$

where $d(x,\alpha)$ denotes the distance between x and the set α with respect to an arbitrary norm d on $\mathbb{R}^d$. The set $\alpha \subset \mathbb{R}^d$ for which the infimum in (1) is attained is called an optimal set of n-means or n-optimal set for μ, and the collection of all n-optimal sets for μ is denoted by $\mathcal{C}_n(\mu)$. Under some suitable restriction $e_n(\mu)$ tends to zero as n tends to infinity. Following [12] we write

$$\hat{e}_n := \hat{e}_n(\mu) = \log e_n(\mu) = \inf\Big\{\int \log d(x,\alpha)d\mu(x) : \alpha \subset \mathbb{R}^d,\ 1 \leq \text{card}(\alpha) \leq n\Big\}.$$

The numbers

$$\overline{D}(\mu) := \limsup \frac{\log n}{-\hat{e}_n(\mu)}, \text{ and } \underline{D}(\mu) := \liminf \frac{\log n}{-\hat{e}_n(\mu)},$$

are called the *upper* and the *lower quantization dimensions* of μ (of order zero) respectively. If $\overline{D}(\mu) = \underline{D}(\mu)$, the common value is called the quantization dimension of μ and is denoted by $D(\mu)$. The quantization dimension measures the speed at which the specified measure of the error tends to zero as n tends to infinity. The quantization dimension with respect to the geometric mean error can be regarded as a limit state of that based on L_r-metrics as r tends to zero (see [12, Lemma 3.5]). The following proposition gives a characterization of the lower and the upper quantization dimensions.

Proposition 1.1. *(See [12, Proposition 4.3].) Let $\underline{D} = \underline{D}(\mu)$ and $\overline{D} = \overline{D}(\mu)$.*

(a) *If $0 \leq t < \underline{D} < s$, then*

$$\lim_{n\to\infty}(\log n + t\hat{e}_n(\mu)) = +\infty, \text{ and } \liminf_{n\to\infty}(\log n + s\hat{e}_n(\mu)) = -\infty.$$

(b) *If $0 \leq t < \overline{D} < s$, then*

$$\limsup_{n\to\infty}(\log n + t\hat{e}_n(\mu)) = +\infty, \text{ and } \lim_{n\to\infty}(\log n + s\hat{e}_n(\mu)) = -\infty.$$

For any $\kappa > 0$, the two numbers $\limsup_n n^{1/\kappa}e_n(\mu)$ and $\liminf_n n^{1/\kappa}e_n(\mu)$ are called the κ*-dimensional upper* and the *lower quantization coefficients* for μ with respect to the geometric mean error. For a Borel probability measure μ on $\mathbb{R}^d$, its Hausdorff dimension $\dim^*_{\mathrm{H}}(\mu)$ is defined by (see [7])

$$\dim^*_{\mathrm{H}}(\mu) := \inf\Big\{\dim_{\mathrm{H}} B : B \text{ is Borel and } \mu(B) = 1\Big\},$$

where $\dim_{\mathrm{H}} B$ denotes the Hausdorff dimension of the set B. Let $\{S_1, S_2, \cdots, S_N\}$ be a finite system of contractive similarity mappings on $\mathbb{R}^d$ with similarity ratios $s_1, s_2, \cdots, s_N$, and $(p_1, p_2, \cdots, p_N)$ be a probability vector with all $p_i > 0$. Then by [14], there exists a unique nonempty compact set E (known as self-similar set) and a unique Borel probability measure μ (known as self-similar measure) with support E, such that the following equations are satisfied:

$$E = \bigcup_{i=1}^N S_i(E), \text{ and } \mu = \sum_{i=1}^N p_i\mu \circ S_i^{-1}.$$