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Covering of spheres by spherical caps and worst-case error for equal weight cubature in Sobolev spaces ☆

J.S. Brauchart [a,b,*], J. Dick [b], E.B. Saff [c], I.H. Sloan [b], Y.G. Wang [b], R.S. Womersley [b]

[a] *Institut für Analysis und Computational Number Theory, Graz University of Technology, Steyrergasse 30, 8010 Graz, Austria*
[b] *School of Mathematics and Statistics, University of New South Wales, Sydney, NSW 2052, Australia*
[c] *Center for Constructive Approximation, Department of Mathematics, Vanderbilt University, Nashville, TN 37240, USA*

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ABSTRACT

We prove that the covering radius of an N-point subset X_N of the unit sphere $\mathbb{S}^d \subset \mathbb{R}^{d+1}$ is bounded above by a power of the worst-case error for equal weight cubature $\frac{1}{N}\sum_{\mathbf{x}\in X_N} f(\mathbf{x}) \approx \int_{\mathbb{S}^d} f \,\mathrm{d}\sigma_d$ for functions in the Sobolev space $\mathbb{W}_p^s(\mathbb{S}^d)$, where σ_d denotes normalized area measure on $\mathbb{S}^d$. These bounds are close to optimal when s is close to d/p. Our study of the worst-case error along with results of Brandolini et al. motivate the definition of Quasi-Monte Carlo (QMC) design sequences for $\mathbb{W}_p^s(\mathbb{S}^d)$, which have previously been introduced only in the Hilbert space setting $p = 2$. We say that a sequence (X_N) of N-point configurations is a QMC-design sequence for $\mathbb{W}_p^s(\mathbb{S}^d)$ with $s > d/p$ provided the worst-case equal weight cubature error for X_N has order $N^{-s/d}$ as $N \to \infty$, a property that holds, in particular, for a sequence of spherical t-designs in which each design has order t^d points. For the case $p = 1$, we deduce that any QMC-design sequence (X_N) for $\mathbb{W}_1^s(\mathbb{S}^d)$ with $s > d$ has the optimal covering property; i.e., the covering radius of X_N has order $N^{-1/d}$ as $N \to \infty$. A significant portion of our effort is devoted to the formulation of the worst-case error in terms of a Bessel kernel, and showing that this kernel satisfies a Bernstein type inequality involving the mesh ratio of X_N. As a consequence we prove that any QMC-design sequence for $\mathbb{W}_p^s(\mathbb{S}^d)$ is also a QMC-design sequence for $\mathbb{W}_{p'}^s(\mathbb{S}^d)$ for all $1 \leq p < p' \leq \infty$ and, furthermore, if (X_N) is a quasi-uniform QMC-design sequence for $\mathbb{W}_p^s(\mathbb{S}^d)$, then it is also a QMC-design sequence for $\mathbb{W}_p^{s'}(\mathbb{S}^d)$ for all $s > s' > d/p$.

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* Corresponding author at: Institut für Analysis und Computational Number Theory, Graz University of Technology, Steyrergasse 30, 8010 Graz, Austria.

E-mail addresses: j.brauchart@tugraz.at (J.S. Brauchart), josef.dick@unsw.edu.au (J. Dick), edward.b.saff@vanderbilt.edu (E.B. Saff), i.sloan@unsw.edu.au (I.H. Sloan), yuguang.e.wang@gmail.com (Y.G. Wang), r.womersley@unsw.edu.au (R.S. Womersley).

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1. Introduction

In this paper we consider covering the unit sphere $\mathbb{S}^d$ in $\mathbb{R}^{d+1}$, $d \geq 1$, with equal sized spherical caps, and establish a connection to equal weight cubature formulas that use the centers of those caps as sampling points for the function. As a corollary, we will show that the optimal order of convergence of the worst-case equal weight cubature error for functions in a suitable Sobolev space implies asymptotically an optimal covering property by spherical caps.

Equal-weight numerical integration In the literature equal weight cubature is often given the name Quasi-Monte Carlo (see Niederreiter [24] for the case of the unit cube). Thus a *Quasi-Monte Carlo (QMC) method* is an equal weight numerical integration formula with *deterministic* node set in contrast to Monte Carlo methods: for a node set $X_N = \{\mathbf{x}_1, \dots, \mathbf{x}_N\} \subset \mathbb{S}^d$, the QMC method

$$\mathrm{Q}[X_N](f) := \frac{1}{N} \sum_{k=1}^{N} f(\mathbf{x}_k)$$

is a natural approximation of the integral

$$\mathrm{I}(f) := \int_{\mathbb{S}^d} f(\mathbf{x}) \mathrm{d}\sigma_d(\mathbf{x})$$

of a given continuous real-valued function f on $\mathbb{S}^d$ with respect to the normalized surface area measure on $\mathbb{S}^d$. A node set X_N is deterministically chosen in a sensible way so as to guarantee "small" error of numerical integration for functions in suitable subfamilies of the class of continuous functions $C(\mathbb{S}^d)$.

A fundamental example of such node sets are *spherical t-designs*[1] $Z_{N_t} \subset \mathbb{S}^d$, $t \geq 1$, introduced in [10]. They define QMC methods that integrate exactly all spherical polynomials of degree $\leq t$:

$$\mathrm{Q}[Z_{N_t}](P) = \mathrm{I}(P), \qquad \deg P \leq t. \tag{1.1}$$

Thus, spherical t-designs yield zero error on polynomial subfamilies of $C(\mathbb{S}^d)$. The definition of spherical t-designs says nothing about the number of points N_t that might be needed. A lower bound on N_t of order t^d was given in [10]. Recently, Bondarenko et al. [4] proved:

Proposition 1.1. *There exists $c_d > 0$ such that to* every $N \geq c_d t^d$ *and $t \geq 1$ there exists an N-point spherical t-design on $\mathbb{S}^d$.*

This key result ensures that spherical t-designs with N_t points of exactly the optimal order t^d exist for every $t \geq 1$ (we write $N_t \asymp t^d$). A sequence (Z_{N_t}) of such designs with optimal order for the number of points has the remarkable property, see [8,14], that

$$|\,\mathrm{Q}[Z_{N_t}](f) - \mathrm{I}(f)| \leq c\, N_t^{-s/d} \, \|f\|_{H^s}$$

for all functions f in a Sobolev space H^s with smoothness index $s > d/2$ and norm $\|\cdot\|_{H^s}$ in the Hilbert space setting. The order of N_t cannot be improved, see [12,13]. This observation motivated the introduction of *QMC-design sequences* for Sobolev spaces H^s in [9]: these are sequences of N-point sets that have the same error behavior as spherical t-designs, but with no polynomial exactness requirement. One purpose of this paper is to provide the extension to general Sobolev spaces.

[1] The symbol X_N is used for general sets of N points on $\mathbb{S}^d$, while Z_{N_t} always refers to a spherical t-design with N_t points.