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A substitution vector-valued integral operator

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ABSTRACT

In this paper we introduce a substitution vector-valued integral operator T_u^φ on $L^2(X)$ associated with the pair (u, φ) and investigate some fundamental properties of T_u^φ by the language of conditional expectation operators.

1. Introduction and preliminaries

Let (X, Σ, μ) be a σ-finite measure space and $\varphi : X \to X$ be a non-singular measurable transformation; i.e. $\mu \circ \varphi^{-1} \ll \mu$. Here the non-singularity of φ guarantees that the composition operator $C_\varphi : f \mapsto f \circ \varphi$ is well defined as a mapping on $L^0(\Sigma)$ where $L^0(\Sigma)$ denotes the linear space of all equivalence classes of Σ-measurable functions on X. Let $h_0 = d\mu \circ \varphi^{-1}/d\mu$ be the Radon–Nikodym derivative. Recall that C_φ is bounded on $L^2(\Sigma)$ if and only if $h_0 \in L^\infty(\Sigma)$ (see [7]). We have the following change of variable formula:

$$\int_{\varphi^{-1}(A)} f \circ \varphi d\mu = \int_A h_0 f d\mu, \quad A \in \Sigma, \ f \in L^1(\Sigma).$$

The support of a measurable function f is defined by $\sigma(f) = \{x \in X : f(x) \neq 0\}$. All comparisons between two functions or two sets are to be interpreted as holding up to a μ-null set. For a sub-σ-finite algebra $\mathcal{A} \subseteq \Sigma$, the conditional expectation operator associated with $\mathcal{A}$ is the mapping $f \to E^{\mathcal{A}} f$, defined for all non-negative f as well as for all $f \in L^p(\Sigma)$, $1 \leq p \leq \infty$, where $E^{\mathcal{A}} f$, by Radon–Nikodym theorem, is the unique $\mathcal{A}$-measurable function satisfying

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$$\int_A f d\mu = \int_A E^{\mathcal{A}} f d\mu, \qquad \forall A \in \mathcal{A}.$$

We recall that $E^{\mathcal{A}} : L^2(\Sigma) \to L^2(\mathcal{A})$ is an orthogonal projection. Throughout this paper, we assume that $\mathcal{A} = \varphi^{-1}(\Sigma)$ and $E^{\varphi^{-1}(\Sigma)} = E$. It is well known that for each non-negative Σ-measurable function f or for each $f \in L^2(\Sigma)$, there exists a Σ-measurable function g such that $E(f) = g \circ \varphi$. We can assume that $\sigma(g) \subseteq \sigma(h_0)$ and there exists only one g with this property. We then write $g = E(f) \circ \varphi^{-1}$ though we make no assumptions regarding the invertibility of φ (see [1]). Let $u \in L^0(\Sigma)$. Thus u is said to be conditionable with respect to E if $u \in \mathcal{D}(E) \subseteq L^0(\Sigma)$, where $\mathcal{D}(E)$ denotes the domain of E. For more details on the properties of $E^{\mathcal{A}}$ see [4,6].

An atom of the measure μ is an element $A \in \Sigma$ with $\mu(A) > 0$, such that for each $B \in \Sigma$, if $B \subset A$ then either $\mu(B) = 0$ or $\mu(B) = \mu(A)$. A measure with no atoms is called non-atomic. We can easily check the following well-known facts (see[8]):

(a) Every σ-finite measure space (X, Σ, μ) can be partitioned uniquely as

$$X = (\cup_{n\in\mathbb{N}} A_n) \cup B, \tag{1.1}$$

where $\{A_n\}_{n\in\mathbb{N}} \subseteq \Sigma$ is a countable collection of pairwise disjoint atoms and B, being disjoint from each A_n, is non-atomic.

(b) Let E be a non-atomic set with $\mu(E) > 0$. Then there exists a sequence of positive disjoint Σ-measurable subsets of E, $\{E_n\}_{n\in\mathbb{N}}$ such that $\mu(E_n) > 0$ for each $n \in \mathbb{N}$ and $\lim_{n\to\infty} \mu(E_n) = 0$.

For a given complex Hilbert space $\mathcal{H}$ with inner product $\langle \cdot, \cdot \rangle$, let $L^2(X, \mathcal{H})$ be the class of all measurable mappings $f : X \to \mathcal{H}$ such that $\|f\|_2^2 := \int_X \|f(x)\|^2 d\mu < \infty$. Let $f, g \in L^2(X, \mathcal{H})$. By using the polar identity, the mapping $x \mapsto \langle f(x), g(x) \rangle$ from X into $\mathbb{C}$ is measurable. It follows that $L^2(X, \mathcal{H})$ is a Hilbert space with inner product

$$(f, g) = \int_X \langle f(x), g(x) \rangle d\mu, \quad f, g \in L^2(X, \mathcal{H}).$$

We shall write $L^2(X)$ for $L^2(X, \mathcal{H})$ when $\mathcal{H} = \mathbb{C}$. Let $u : X \to \mathcal{H}$ be a mapping. We say that u is weakly measurable if for each $h \in \mathcal{H}$ the mapping $x \mapsto \langle u(x), h \rangle$ from X into $\mathbb{C}$ is measurable. We will denote this map by $\langle u, h \rangle$.

The aim of this article is to carry some of the results obtained for the weighted composition operators in [2,3,7] to a substitution vector-valued integral operator. In this paper, first we consider some basic properties of substitution vector-valued integral operators and then we give some necessary and sufficient conditions for boundedness, compactness and semi-Fredholmness of these type operators.

2. The main results

Definition 2.1. Let $u : X \to \mathcal{H}$ be a weakly measurable function. We say that u is a weakly bounded function if for some $B \geq A > 0$,

$$\sqrt{A}\|h\| \leq \|\langle u, h \rangle\|_2 \leq \sqrt{B}\|h\|, \qquad \forall h \in \mathcal{H}. \tag{2.1}$$

In the same way, u is said to be semi-weakly bounded function if u only satisfies the right hand side of the above inequalities.