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# Closed range composition operators on Hilbert function spaces

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ABSTRACT

We show that a Carleson measure satisfies the reverse Carleson condition if and only if its Berezin symbol is bounded below on the unit disk $\mathbb{D}$. We provide new necessary and sufficient conditions for the composition operator to have closed range on the Bergman space. The pull-back measure of area measure on $\mathbb{D}$ plays an important role. We also give a new proof in the case of the Hardy space and conjecture a condition in the case of the Dirichlet space.



## 1. Introduction

Let $\varphi$ be an analytic self-map of the unit disk $\mathbb{D}$. The *composition operator with symbol* $\varphi$ is defined by

$$C_\varphi(f) = f \circ \varphi \,,$$

for any function $f$ that is analytic on $\mathbb{D}$. Littlewood in 1925 proved a subordination principle, which in operator theory language, says that composition operators are bounded in the *Hardy space* $H^2$, the Hilbert space of analytic functions on $\mathbb{D}$ with square summable power series coefficients. This is the first setting by which many properties of the composition operator such as boundedness, compactness, and closed range have been studied. It is natural to study these properties on other function spaces.

Let $\mathcal{H}$ be a Hilbert space of analytic functions on $\mathbb{D}$ with inner product $\langle \cdot\,, \cdot \rangle$. We say that $\mathcal{H}$ is a *Hilbert function space* if all point evaluations are bounded linear functionals. By the Riesz representation theorem, for each $z \in \mathbb{D}$, there exists a unique element $K_z$ of $\mathcal{H}$, called the *reproducing kernel* at $z$, such that for each $f \in \mathcal{H}$, $f(z) = \langle f, K_z \rangle$. For each $z \in \mathbb{D}$ we have

$$|f(z)| \le \|f\| \, \|K_z\| = \|f\| \, K_z(z)^{1/2}, \tag{1}$$

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where $\|\cdot\|$ denotes the norm in $\mathcal{H}$. In particular for each $z, w \in \mathbb{D}$ we have

$$| < K_w, K_z > | = |K_w(z)| \le K_w(w)^{1/2}\, K_z(z)^{1/2}\,. \tag{2}$$

For each $w \in \mathbb{D}$, the *normalized reproducing kernel* in $\mathcal{H}$ is

$$k_w(z) = \frac{K_w(z)}{||K_w||}\,, \quad z \in \mathbb{D}\,. \tag{3}$$

If $C_\varphi$ is a bounded operator on $\mathcal{H}$, then by Theorem 1.4 in [11] we have

$$C^*_\varphi(K_z) = K_{\varphi(z)} \tag{4}$$

and hence

$$||K_{\varphi(z)}|| \le ||C_\varphi||\, ||K_z||\,. \tag{5}$$

Cima, Thomson and Wogen in [9] were the first to study closed range composition operators in $H^2$. Their results were in terms of the boundary behavior of $\varphi$. Next, Zorboska in [27] studied closed range composition operators in $H^2$ and in the weighted Hilbert Bergman space in terms of properties of $\varphi$ inside $\mathbb{D}$. Since then several authors studied this problem in different Banach spaces of analytic functions, see for example [14, 2–4,18,25].

In this paper we continue the study of closed range composition operators on the Bergman space $A^2$, the Hardy space $H^2$ and the Dirichlet space $\mathcal{D}$. We will define and discuss properties of these spaces as well as other preliminary work in Section 2. In Section 3 we develop general machinery that can be useful in studying closed range composition operators in any Hilbert function space $\mathcal{H}$ with reproducing kernel $K_z$. Given $\varepsilon > 0$, let

$$\Lambda_\varepsilon = \Lambda_\varepsilon(\mathcal{H}) = \left\{ z \in \mathbb{D} : ||K_{\varphi(z)}|| \;>\; \varepsilon ||K_z|| \right\}$$

and $G_\varepsilon = G_\varepsilon(\mathcal{H}) = \varphi(\Lambda_\varepsilon)$. In Section 3 we give a necessary condition for $G_\varepsilon$ to intersect each pseudohyperbolic disk in $\mathbb{D}$, see Proposition 3.2. It is useful in our results in Section 4 and in Section 6.

In Section 4 we build on known results about closed range composition operators on $A^2$ provided in [2] and [25]. It is well known that $\mu$ is a Carleson measure on $\mathbb{D}$ if and only if the Berezin symbol of $\mu$ is bounded; our first main result in Section 4 is an analog of this for Carleson measures that satisfy the reverse Carleson condition, see Theorem 4.1. We use this to provide necessary and sufficient conditions for the pull-back measure of area measure on $\mathbb{D}$ to satisfy the reverse Carleson condition, see Theorems 4.2 and 4.3.

Akeroyd and Ghatage in [2] showed that $C_\varphi$ is closed range in $\mathcal{H} = A^2$ if and only if for some $\varepsilon > 0$ the set $G_\varepsilon$ above satisfies the reverse Carleson condition. This means that $G_\varepsilon$ intersects every pseudohyperbolic disk, of some fixed radius $r \in (0, 1)$, in a set that has area comparable to the area of each pseudohyperbolic disk. In Theorem 4.4 we show that in fact this is equivalent to $G_\varepsilon$ merely having non-empty intersection with each pseudohyperbolic disk. Moreover we provide an analog of [9, Theorem 2] in $A^2$ in terms of the pull-back measure of normalized area measure on $\mathbb{D}$; we also show that $C_\varphi$ is closed range on $A^2$ if and only if for all $w \in \mathbb{D}$, $||C_\varphi K_w|| \asymp ||K_w||$. Lastly we provide a condition that makes it easy to check whether $C_\varphi$ is closed range on $A^2$, see (e) of Theorem 4.4.

In Section 5 we revisit closed range composition operators on $H^2$. The first main result of this section is not new. It is a combination of [25, Theorem 5.4] and a result in Luery's thesis [18, Theorem 5.2.1]. Our new short proof uses pseudohyperbolic disks. Zorboska proved in [27] that for univalent symbols, $C_\varphi$ is closed range on $A^2$ if and only if it is closed range on $H^2$. We extend Zorboska's result to include all symbols $\varphi \in \mathcal{D}$. Akeroyd and Ghatage show in [2] that the only univalent symbols that give rise to a closed range $C_\varphi$