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A multiple Hilbert-type discrete inequality with a new kernel and best possible constant factor ☆

Bing He [a], Qiru Wang [b,*]

[a] Department of Mathematics, Guangdong University of Education, Guangzhou 510303, PR China
[b] School of Mathematics and Computational Science, Sun Yat-sen University, Guangzhou 510275, PR China

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ABSTRACT

By introducing a distinctive kernel and using the way of weight functions, a brand new multiple Hilbert-type discrete inequality with a best constant factor is presented. The operator expressions, equivalent forms, reverse inequalities and particular cases are discussed.

1. Introduction

Suppose that $p > 1, (1/p) + (1/q) = 1, a_m, b_n \geq 0, a = \{a_m\}_{m=1}^{\infty} \in l^p, b = \{b_n\}_{n=1}^{\infty} \in l^q, ||a||_p = \{\sum_{m=1}^{\infty} a_m^p\}^{1/p} > 0, ||b||_q > 0$, the well-known Hardy–Hilbert's discrete inequality is given by [1, p. 226]:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n} < \frac{\pi}{\sin(\pi/p)}||a||_p||b||_q, \tag{1}$$

and an equivalent form is

$$\left\{\sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty}\frac{a_m}{m+n}\right)^p\right\}^{1/p} < \frac{\pi}{\sin(\pi/p)}||a||_p, \tag{2}$$

where the constant factor $\pi/\sin(\pi/p)$ is the best possible. Define Hardy–Hilbert's discrete operator $T : l^p \to l^p$ by: $(Ta)(n) := \sum_{m=1}^{\infty}\frac{a_m}{m+n}$, where $a = \{a_m\}_{m=1}^{\infty} \in l^p, n \in \mathbf{N}$. Then (2) can be written as

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* Corresponding author.
E-mail addresses: hzs314@163.com (B. He), mcswqr@mail.sysu.edu.cn (Q. Wang).

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$||Ta||_p < [\pi/\sin(\pi/p)]\,||a||_p$ and $||T|| \leq \pi/\sin(\pi/p)$. Since the constant factor in (2) is the best possible, we have $||T|| = \pi/\sin(\pi/p)$.

Inequalities (1) and (2) have been studied extensively, which are important in analysis and applications [3,10,11,14–16]. Hilbert-type inequalities may be classified into several types (integral, discrete and half-discrete, etc.). On discrete inequalities, by introducing an independent parameter λ $(2-\min\{p,q\} < \lambda \leq 2)$, Yang [17] gave an extension of (1) in 2002 as follows:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{(m+n)^{\lambda}} < B\left(\frac{p+\lambda-2}{p}, \frac{q+\lambda-2}{q}\right)||a||_{p,\xi}||b||_{q,\xi}, \tag{3}$$

where $B(u,v)$ is the Beta function, $\xi(n) = n^{1-\lambda}, ||a||_{p,\xi} := \{\sum_{n=1}^{\infty}\xi(n)a_n^p\}^{1/p} > 0, ||b||_{q,\xi} > 0$. In the same year, another similar form was established by Yang [13]:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m^{\mu}+n^{\mu}} < \frac{\pi}{\mu\sin(\pi/p)}||a||_{p,\phi}||b||_{q,\psi}, \tag{4}$$

where $0 < \mu \leq \min\{p,q\}, \phi(n) = n^{(p-1)(1-\mu)}, \psi(n) = n^{(q-1)(1-\mu)}, ||a||_{p,\phi} := \{\sum_{n=1}^{\infty}\phi(n)a_n^p\}^{1/p} > 0$, $||b||_{q,\psi} > 0$.

It is significant to generalize Hilbert-type inequalities into multiple versions. In recent years, some results have been obtained [4–8,12,18,19]. In 2009, the following multiple Hilbert-type discrete inequality had been studied by Yang [15, Example 9.1.1]:

If $n \in \mathbf{N}\backslash\{1\}, p_i, r_i > 1, \sum_{i=1}^{n}\frac{1}{p_i} = \sum_{i=1}^{n}\frac{1}{r_i} = 1, 0 < \lambda \leq \min_{1\leq i\leq n}\{r_i\}$, $0 < \sum_{m_i=1}^{\infty} m_i^{p_i(1-\lambda/r_i)-1} \times \left(a_{m_i}^{(i)}\right)^{p_i} < \infty$ $(i = 1,\cdots,n)$, then

$$\sum_{m_n=1}^{\infty}\cdots\sum_{m_1=1}^{\infty}\frac{1}{(\sum_{i=1}^{n}m_i)^{\lambda}}\prod_{i=1}^{n}a_{m_i}^{(i)}$$
$$< \frac{1}{\Gamma(\lambda)}\prod_{i=1}^{n}\Gamma\left(\frac{\lambda}{r_i}\right)\left\{\sum_{m_i=1}^{\infty} m_i^{p_i(1-\frac{\lambda}{r_i})-1}\left(a_{m_i}^{(i)}\right)^{p_i}\right\}^{1/p_i}, \tag{5}$$

where the constant factor $\frac{1}{\Gamma(\lambda)}\prod_{i=1}^{n}\Gamma\left(\frac{\lambda}{r_i}\right)$ is the best possible. It is not an easy job to find new forms of such inequalities. Exploiting these multiple inequalities will greatly increase the scope of our work. This paper will include a completely new multiple discrete Hilbert-type inequality with an interesting kernel.

The setup of this paper is as follows. In Section 2, a number of basic lemmas are obtained. To calculate the constant factor, we herein utilize mathematical induction to do an excellent computing. To prove the best of the constant factor, we apply some analytical skills to do a good estimate for some formula. Section 3 presents our main results, the techniques that will be used in the proof are mainly based on classical real analysis, especially on the well-known Hölder's inequality. In addition, the equivalent forms, operator expressions as well as some reverse inequalities are also considered.

2. Basic lemmas

Lemma 1. *(See [15, (9.1.1)].) If $n \in \mathbf{N}\backslash\{1\} = \{2,3,4,5,\cdots\}, p_i \neq 0, 1 (i = 1,2,\cdots,n), \sum_{i=1}^{n}\frac{1}{p_i} = 1$, then*

$$\prod_{i=1}^{n}\left[m_i^{p_i-1}\prod_{j=1(j\neq i)}^{n}m_j^{-1}\right]^{\frac{1}{p_i}} = 1. \tag{6}$$