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Dynamics of the birational maps arising from F_0 and dP_3 quivers

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ABSTRACT

The dynamics of the maps associated to F_0 and dP_3 quivers is studied in detail. We show that the corresponding reduced symplectic maps are conjugate to globally periodic maps by providing explicit conjugations. The dynamics in $\mathbb{R}^N_+$ of the original maps is obtained by lifting the dynamics of these globally periodic maps and the solution of the discrete dynamical systems generated by each map is given. A better understanding of the dynamics is achieved by considering first integrals. The relationship between the complete integrability of the globally periodic maps and the dynamics of the original maps is explored.

1. Introduction

We study the dynamics of the discrete dynamical systems (DDS) given by the iterates of the following maps

$$\varphi(x_1, x_2, x_3, x_4) = \left(x_3, x_4, \frac{x_2^2 + x_3^2}{x_1}, \frac{x_1^2 x_4^2 + (x_2^2 + x_3^2)^2}{x_1^2 x_2}\right) \tag{F_0}$$

$$\varphi(x_1, x_2, \dots, x_6) = \left(x_3, x_4, x_5, x_6, \frac{x_2 x_4 + x_3 x_5}{x_1}, \frac{x_1 x_4 x_6 + x_2 x_4 x_5 + x_3 x_5^2}{x_1 x_2}\right) \tag{dP_3}$$

in $\mathbb{R}^4_+$ and $\mathbb{R}^6_+$, respectively.

These maps arise in the context of the theory of cluster algebras associated to quivers (oriented graphs) satisfying a mutation-periodicity property. In particular, the maps (F_0) and (dP_3) are associated to quivers appearing in quiver gauge theories associated to the complex cones over the Hirzebruch zero and del Pezzo 3 surfaces respectively (see [9,12,7]). For this reason, the quivers in Fig. 2 of [7] and their associated maps

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will be called F_0 and dP_3. These quivers have 4 and 6 nodes respectively and their mutation-period is equal to 2. We refer to [7] for the construction of the maps (F_0) and (dP_3) from the respective quivers.

As shown in [12] any mutation-periodic quiver gives rise to a birational map φ whose iterates define a system of k difference equations, where k is the (mutation) period of the quiver. Moreover any quiver with N nodes and no loops nor two cycles is represented by an $N \times N$ integer skew-symmetric matrix which defines a log presymplectic form ω. This presymplectic form is invariant under the associated map φ (i.e. $\varphi^*\omega = \omega$) if and only if the quiver is mutation-periodic [7].

Our study of the dynamics of the maps (F_0) and (dP_3) relies on a result obtained by the first and last authors in [7]. Notably, any birational map associated to a mutation-periodic quiver of period k is semiconjugate to a symplectic map $\hat{\varphi} : \mathbb{R}^p_+ \to \mathbb{R}^p_+$ where p is the rank of the matrix representing the quiver. As the matrices representing the F_0 and dP_3 quivers have rank equal to 2, the reduced maps of (F_0) and (dP_3) are 2-dimensional. In [7], using one of the approaches developed there, reduced maps of (F_0) and of (dP_3) preserving the symplectic form $\omega_0 = \frac{dx \wedge dy}{xy}$ were obtained.

The maps (F_0) and (dP_3) have a rather complicated expression and it seems that there is no simple approach to their dynamics. As the reduced maps $\hat{\varphi}$ are 2-dimensional it is then natural to begin with the study of the dynamics of these maps. We prove that the maps $\hat{\varphi}$ are globally periodic by exhibiting explicit conjugations between them and two simple globally periodic maps ψ (cf. Theorem 1). As a consequence, the original maps φ in (F_0) and (dP_3) are semiconjugate to the globally periodic maps ψ and their dynamics can be studied by lifting the dynamics of the respective map ψ.

The maps ψ corresponding to the maps (F_0) and (dP_3), only have a fixed point and any other point has period 4 and period 6, respectively. We show in Theorems 2, 3 and 4 that the fixed point of ψ lifts to an algebraic variety invariant under φ and that the lift of any m-periodic point, with $m > 1$, gives rise to m algebraic varieties of codimension 2, which are mapped cyclically one into the other by the map φ and are invariant under the map $\varphi^{(m)}$. Moreover, Theorem 3 and Theorem 4 also provide the explicit solution of the DDS generated by the maps (F_0) and (dP_3). In the case of the map (dP_3) we are even able to further confine the orbits to algebraic varieties of codimension 4 by using first integrals (cf. Proposition 2 and Corollary 1).

The key property behind the study of the dynamics of the maps (F_0) and (dP_3) is the global periodicity of their reduced maps. It is well known that globally periodic maps are completely integrable and independent first integrals can be obtained using, for instance, the techniques described in [5]. This allows us to produce independent sets of (lifted) first integrals of (F_0) and of (dP_3). It seems natural to relate the invariant varieties already obtained to the common level sets of the lifted first integrals. This is done in Proposition 3 for a particular choice of lifted first integrals of the map (F_0).

We note that there are some studies in the literature of the (reduced) dynamics of maps arising from quivers with mutation period equal to 1 (see [10] and [11]). However, to the best of our knowledge this is the first time that the dynamics of maps associated to higher periodic quivers is fully described. It is also worth noting that in the referred studies of maps from 1-periodic quivers, the reduced symplectic maps do not exhibit the global periodicity property of the symplectic maps studied in this paper.

The organization of the paper is as follows. In Section 2 we provide the necessary background and collect some results obtained in [7] relevant to the study to be performed. The following section is devoted to the dynamics of the reduced maps where we show that they are globally periodic. In Section 4 we completely describe the dynamics of the maps (F_0) and (dP_3) by lifting the dynamics of the corresponding globally periodic maps. The last section is devoted to the existence of first integrals of (F_0) and (dP_3). First, we use first integrals to obtain a more specific description of the dynamics of the map (dP_3). Second, we explore the relationship between the complete integrability of globally periodic maps and the study of the dynamics of (F_0) and (dP_3) performed in the previous sections.