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Multiply warped products with a quarter-symmetric connection

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ABSTRACT

In this paper, we study the Einstein warped products and multiply warped products with a quarter-symmetric connection. We also study warped products and multiply warped products with a quarter-symmetric connection with constant scalar curvature. Then we apply our results to generalized Robertson–Walker space–times with a quarter-symmetric connection and generalized Kasner space–times with a quarter-symmetric connection.

1. Introduction

The (singly) warped product $B \times_b F$ of two pseudo-Riemannian manifolds (B, g_B) and (F, g_F) with a smooth function $b : B \to (0, \infty)$ is a product manifold of form $B \times F$ with the metric tensor $g = g_B \oplus b^2 g_F$. Here, (B, g_B) is called the base manifold and (F, g_F) is called the fiber manifold and b is called the warping function. The concept of warped products was first introduced by Bishop and O'Neill [2] to construct examples of Riemannian manifolds with negative curvature. In [3], F. Dobarro and E. Dozo had studied the problem of showing when a Riemannian metric of constant scalar curvature can be produced on a product manifolds by a warped product construction from the viewpoint of partial differential equations and variational methods. In [5], Ehrlich, Jung and Kim got explicit solutions to warping function to have a constant scalar curvature for generalized Robertson–Walker space–times. In [1], explicit solutions were also obtained for the warping function to make the space–time as Einstein when the fiber is also Einstein.

One can generalize (singly) warped products to multiply warped products. A multiply warped product (M, g) is the product manifold $M = B \times_{b_1} F_1 \times_{b_2} F_2 \cdots \times_{b_m} F_m$ with the metric $g = g_B \oplus b_1^2 g_{F_1} \oplus b_2^2 g_{F_2} \cdots \oplus b_m^2 g_{F_m}$, where for each $i \in \{1, \cdots, m\}$, $b_i : B \to (0, \infty)$ is smooth and (F_i, g_{F_i}) is a pseudo-Riemannian

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manifold. In particular, when $B = (c, d)$, the metric $g_B = -dt^2$ is negative and (F_i, g_{F_i}) is a Riemannian manifold, we call M as the multiply generalized Robertson–Walker space–time.

Singly warped products have a natural generalization. A twisted product (M, g) is a product manifold of form $M = B \times_b F$, with a smooth function $b : B \times F \to (0, \infty)$, and the metric tensor $g = g_B \oplus b^2 g_F$. In [6], it was shown that mixed Ricci-flat twisted products could be expressed as warped products. As a consequence, any Einstein twisted products are warped products. Similar to the definition of multiply warped product, a multiply twisted product (M, g) is a product manifold of the form $M = B \times_{b_1} F_1 \times_{b_2} F_2 \cdots \times_{b_m} F_m$ with the metric $g = g_B \oplus b_1^2 g_{F_1} \oplus b_2^2 g_{F_2} \cdots \oplus b_m^2 g_{F_m}$, where for each $i \in \{1, \cdots, m\}$, $b_i : B \times F_i \to (0, \infty)$ is smooth. So in this paper, we define the multiply twisted products as generalizations of twisted products and multiply warped products.

The definition of a semi-symmetric metric connection was given by H. Hayden in [8]. In 1970, K. Yano [15] considered a semi-symmetric metric connection and studied some of its properties. Then in 1975, Golab [7] introduced the idea of a quarter-symmetric linear connection in differentiable manifold which is a generalization of semi-symmetric connection. A linear connection ∇ on an n-dimensional Riemannian manifold (M, g) is called a quarter-symmetric connection if its torsion tensor T of the connection ∇ satisfies $T(X, Y) = \pi(Y)\phi X - \pi(X)\phi Y$, where π is a 1-form and ϕ is a $(1, 1)$ tensor field. In particular, if $\phi(X) = X$, then the quarter-symmetric connection reduces to the semi-symmetric connection.

In [4], Dobarro and Ünal studied Ricci-flat and Einstein–Lorentzian multiply warped products and considered the case of having constant scalar curvature for multiply warped products and applied their results to generalized Kasner space–times. In [10], S. Sular and C. Özgür studied warped product manifolds with a semi-symmetric metric connection, they computed curvature of semi-symmetric metric connection and considered Einstein warped product manifolds with a semi-symmetric metric connection. In [11], they studied warped product manifolds with a semi-symmetric non-metric connection. In [14], we considered multiply warped products with a semi-symmetric metric connection, then apply our results to generalized Robertson–Walker spacetimes with a semi-symmetric metric connection and generalized Kasner spacetimes with a semi-symmetric metric connection. In [13], we studied curvature of multiply warped products with a semi-symmetric non-metric connection. In this paper, we will generalize our result to warped and multiply warped products with a special quarter-symmetric connection which satisfies equations (2.5) and (2.6) in [12]. This special quarter-symmetric connection is defined by equation (3). All the work we do is about it.

This paper is arranged as follows. In Section 2, we get a special quarter-symmetric connection and its curvature, then give the formula of the Levi-Civita connection and curvature of singly warped and multiply twisted product. In Section 3, we first compute curvature of a singly warped product with this quarter-symmetric connection, then study the generalized Robertson–Walker space–times with this quarter-symmetric connection. In Section 4, firstly we compute curvature of multiply twisted products with this quarter-symmetric connection, secondly we study the special multiply warped product with this quarter-symmetric connection, finally we consider the generalized Kasner space–times with this quarter-symmetric connection.

2. Preliminaries

Let M be a Riemannian manifold with Riemannian metric g. A linear connection $\overline{\nabla}$ on a Riemannian manifold M is called a quarter-symmetric connection if the torsion tensor T of the connection $\overline{\nabla}$

$$T(X, Y) = \overline{\nabla}_X Y - \overline{\nabla}_Y X - [X, Y] \quad (1)$$

satisfies

$$T(X, Y) = \pi(Y)\phi X - \pi(X)\phi Y \quad (2)$$