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On the uniqueness and stability of an inverse problem in photo-acoustic tomography

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This article deals with the uniqueness and stability of the solution of a problem of optimal control related to the photo-acoustic tomography process. We prove stability results of the optimal solution with respect to the source and to the observation data and we compute the corresponding derivatives.

1. Introduction

In this paper we consider a differential system arising in photo-acoustic tomography. We refer [12] to get a complete description of the model. Let us briefly mention that we deal with two coupled partial differential equations that describes the light intensity (fluence) behavior inside a body that is excited by a laser (pulsed) source and the acoustic pressure wave which is generated by this excitation. The authors of [12] have investigated the model and obtained an optimal control formulation to recover some parameters of interest, namely the absorption and diffusion coefficients (μ, D).

We want to address the optimal solution sensitivity with respect to the source and the observation data that appears in the wave equation. For that purpose, in a first step we assume that the diffusion coefficient is constant (and for sake of simplicity equal to 1).

In this work, we prove uniqueness and stability results provided that the coercivity constant α of the cost functional J, given by (2.4), is large enough.

From this point of view, the result is similar to the one of [10]. Other results of uniqueness and stability, in the context of the photo-acoustic, have been obtained in [2,3,7–9,11]. For example, in [8,9] the authors obtained uniqueness and stability results under the assumption that the function $H(x) := \Gamma(x)\mu(x)I_\mu(x)$

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is known and the absorption and diffusion coefficients are smooth enough. Moreover, they did not consider the whole process which couples lightning and acoustic wave equations.

The stability of optimal controls have also studied in [13,18,20,22,23] in other settings.

The paper is structured as follows. In Section 2, we recall the problem setting and preliminary results. Section 3 is devoted to stability and uniqueness properties. In Section 4, we compute the derivative of the optimal control with respect to the source giving a characterization. We also study the stability of the optimal solution with respect to the observation. We end the paper with conclusions and a few words on future work.

2. Problem setting

2.1. Photo-acoustic modelling

Recall photo-acoustic tomography (PAT) principle: tissues to be imaged are illuminated by a laser (the source). This energy is converted into heat creating a thermally induced pressure jump that propagates as a sound wave, which can be detected. The fluence rate I_μ, that is the average of the luminous intensity in all the directions, satisfies the diffusion equation (see [1,5,12])

$$\begin{cases} \dfrac{1}{c}\dfrac{\partial I_\mu}{\partial t}(t,x) + \mu(x) I_\mu(t,x) - \Delta I_\mu(t,x) = S(t,x) & \text{in } (0,T)\times\Omega \\ I_\mu(t,x) = 0 & \text{on } (0,T)\times\partial\Omega \\ I_\mu(0,x) = 0 & \text{in } \Omega. \end{cases} \tag{2.1}$$

where c is the speed of light, S is the incident light source, μ is the *absorption coefficient*, and $T > 0$ is the duration of the acquisition process.

Here, Ω stands for the part of the body where the diffusion approximation is relevant and the diffusion coefficient has been set to 1 for simplicity. It is an open subset of $\mathbb{R}^d$ ($d \geq 2$) of class C^2. For a fixed $T > 0$, we will often denote $Q := (0,T)\times\Omega$.

The acoustic wave that is generated is described via the pressure p_μ that satisfies (up to the change of variables: $p \mapsto \int_0^t p(s)\,ds$):

$$\begin{cases} \dfrac{\partial^2 p_\mu}{\partial t^2}(t,x) - \operatorname{div}(v_s^2 \nabla p_\mu)(t,x) = \mathbb{1}_\Omega(x)\Gamma(x)\mu(x) I_\mu(t,x) & \text{in } (0,T)\times\mathcal{B} \\ p_\mu(t,x) = 0 & \text{on } (0,T)\times\partial\mathcal{B} \\ p_\mu(0,x) = \dfrac{\partial p_\mu}{\partial t}(0,x) = 0 & \text{in } \mathcal{B}. \end{cases} \tag{2.2}$$

Here, the *Grueneisen coefficient* Γ, coupling the energy absorption to the thermal expansion, is assumed to be known. In the sequel we assume that Γ has compact support in Ω so that $\Gamma \mathbb{1}_\Omega = \Gamma$ and that the speed of sound v_s is known and satisfies $v_s \in [v_s^{\min}, v_s^{\max}]$, with $v_s^{\min} > 0$. The ball $\mathcal{B}$ is the domain where the wave propagates. It includes Ω and it has to be bounded in view of numerical simulations. It is large enough to assume that there is no reflected wave before time T.

The absorption coefficient μ is the parameter we want to study. We assume that

$$\mu \in \mathcal{U}_{ad} = \{\mu \in L^\infty(\mathcal{B}) \mid \mu \in [\mu^{\min}, \mu^{\max}] \text{ a.e. in } \mathcal{B}\}, \tag{2.3}$$

where $0 < \mu^{\min} < \mu^{\max}$ are positive real numbers.

The photo-acoustic tomography model is completely described by the coupling of equations (2.1) and (2.2), where I_μ is extended by 0 on $\mathcal{B} \setminus \Omega$. Here S is the incident light source that we assume in $L^2(Q)$.

We first recall the results of [12] (for $D = 1$).