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On geometry of cones and some applications

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ABSTRACT

In this work we prove that in any normed space, the origin is a denting point of a pointed cone if and only if it is a point of continuity for the cone and the closure of the cone in the bidual space with respect to the weak* topology is pointed. Other related results and consequences are also stated. For example, a criterion to know whether a cone has a bounded base, an unbounded base, or does not have any base; and a result on the existence of super efficient points in weakly compact sets.

1. Introduction

The notion of denting point goes back to the early studies of sets with the Radon–Nikodým property in [3]. It has also been applied to renorming theory (e.g. [8] and the references therein) and to optimization [6]. The notion of point of continuity is a generalization of the former one. It was initially used to provide a geometric proof of the Ryll-Nardzewski fixed point theorem in [16]. Later on, it was used for geometric purposes in [3], and it was applied to optimization in [9]. B.L. Lin, P.K. Lin, and S. Troyanski showed in [15] that both notions become equivalent at extreme points of closed, convex, and bounded subsets of Banach spaces (see also [21, Proposition 3.3]).

Regarding cones, X.H. Gong asked in [9, Conclusions] a question which can be restated in the following way: *The property that the origin in a normed space be a point of continuity for a closed and pointed cone, is really weaker than that the origin be a denting point of the cone?* (The original statement was stated in terms of bounded bases instead of denting points.) Later on, A. Daniilidis asked negatively such a question (into the frame of Banach spaces) noting the following consequence of the theorem of Lin–Lin–Troyanski

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[6, Corollary 2]: *given a closed and pointed cone C in a Banach space X, the origin (0_X for short) is a denting point of C if and only if it is a point of continuity for C*. In addition, the former characterization allowed Daniilidis to prove the equivalence (into the frame of Banach spaces) between two density results of Arrow, Barankin and Blackwell's type, one due to M. Petschke [18, Corollary 4.2] and another due to Gong [9, Theorem 3.2 (a)].

Daniilidis' characterization [6, Corollary 2] is not true for non-closed cones, as Example 1.5 in Section 1.2 shows. Thus, the answer to Gong's question is positive for non-closed cones. In this line, C. Kountzakis and I.A. Polyrakis showed the following result [14, Theorem 4]: *in any normed space X such that the set of quasi-interior positive elements of X^* is non-empty, 0_X is a denting point of a pointed cone C if and only if it is a point of continuity for C*. The former characterization provides a partial answer to Gong's question in the context of non-closed cones. In addition, it has applications in the theory of Pareto optimization, see [14].

In this work, we continue the research line of [14] and prove the following: *in any normed space X, 0_X is a denting point of a pointed cone C if and only if it is a point of continuity for C and the closure of the cone in the bidual space with respect to the weak* topology is pointed*. It corresponds to the equivalence between (i) and (iii) in Theorem 1.1 below. Let us note that we have changed the assumption in Kountzakis and Polyrakis' theorem [14, Theorem 4] which affects to the whole X^*, by another which only affects to the particular cone we are considering. Our characterization also provides a partial answer to Gong's question in the context of non-closed cones. On the other hand, if X is reflexive, then the closure of the cone in the bidual space with respect to the weak* topology coincides with its closure with respect to (X, weak). Moreover, using Mazur's theorem, it is easily seen that the last set is equal to the closure of C in $(X, \|\cdot\|)$. Thus, for reflexive Banach spaces, our characterization is equivalent to Daniilidis' characterization [6, Corollary 2]. Then, in some way, our characterization can be interpreted as a generalization of [6, Corollary 2] for normed spaces. Some consequences and other related results are also stated and proved in this manuscript. They are stated in Section 1.2. In the following subsection we have compiled the definitions of most of the notions which appear in the work.

1.1. Notation and main definitions

We will denote by X a normed space, by $\|\cdot\|$ the norm of X, by X^* the dual space of X, by $\|\cdot\|_*$ the norm of X^*, by 0_X the origin of X, and by $\mathbb{R}_+$ the set of non-negative real numbers. A non-empty convex subset C of X is called a *cone* if $\alpha C \subset C$, $\forall \alpha \in \mathbb{R}_+$. In what follows, $C \subset X$ stands for a cone. C is called *pointed* if $C \cap (-C) = \{0_X\}$. The cone

$$C^* := \{f \in X^* \colon f(c) \geq 0\,, \forall c \in C\},$$

is called the *dual cone* for C, and the set

$$C^\# := \{f \in X^* \colon f(c) > 0\,, \forall c \in C \setminus \{0_X\}\},$$

the *quasi-relative interior* of the dual cone for C or *the set of all strictly positive functionals*. The interior of C^*, $\operatorname{Int} C^*$, is contained in $C^\#$ and $\operatorname{Int} C^* = C^\#$ whenever the first one is non-empty. Any $c \in C$ is said to be a *quasi-interior point of X* or *a quasi-interior positive element of X* if $\overline{\cup_{n\in\mathbb{N}}[-nc, nc]} = X$, where $[-nc, nc]$ is the order interval (given by the cone order) $\{x \in X \colon -nc \leq x \leq nc\}$. The set of all quasi-interior points is denoted by $\operatorname{qi} C$. If C has non-empty interior, then the concepts of interior point of C and quasi-interior point of X coincide [17]. Besides, $\operatorname{qi} C$ is either empty or dense in C. In general $\operatorname{qi} C^* \subset C^\#$, and $\operatorname{qi} C^* = C^\#$ in the context of normed lattices [1]. A non-empty convex subset B of C is called a *base* for C, if $0 \notin \overline{B}$ and each element $c \in C \setminus \{0\}$ has a unique representation of the form $c = \lambda b$,