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Basic and bibasic identities related to divisor functions

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ABSTRACT

Using basic hypergeometric functions and partial fraction decomposition we give a new kind of generalization of identities due to Uchimura, Dilcher, Van Hamme, Prodinger, and Chen-Fu related to divisor functions. An identity relating Lambert series to Eulerian polynomials is also proved.

1. Introduction

The divisor function $\sigma_m(n)$ for a natural number n is defined as the sum of the mth powers of the (positive) divisors of n, i.e., $\sigma_m(n) = \sum_{d|n} d^m$. Throughout this paper, we assume that $|q| < 1$. The generating function of $\sigma_m(n)$ has an explicit Lambert series expansion (see [5]):

$$\sum_{n=1}^{\infty} \sigma_m(n) q^n = \sum_{n=1}^{\infty} \frac{n^m q^n}{1-q^n}. \tag{1.1}$$

In 1981, Uchimura [24] rediscovered an identity due to Kluyver [19] (see also Dilcher [9]):

$$\sum_{k=1}^{\infty} (-1)^{k-1} \frac{q^{\binom{k+1}{2}}}{(q;q)_k (1-q^k)} = \sum_{k=1}^{\infty} \frac{q^k}{1-q^k}, \tag{1.2}$$

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where $(a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1})$ for $n \geqslant 0$. Since then, many authors have given different generalizations of (1.2) (see [2,4,9,11,12,14,16–18,21,22,24–26,28]). For example, Van Hamme [26] gave a finite form of (1.2) as follows:

$$\sum_{k=1}^{n}(-1)^{k-1}\begin{bmatrix} n \\ k \end{bmatrix}_q \frac{q^{\binom{k+1}{2}}}{1-q^k} = \sum_{k=1}^{n}\frac{q^k}{1-q^k}, \tag{1.3}$$

where the q-binomial coefficients $\begin{bmatrix} n \\ k \end{bmatrix}_q$ are defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}.$$

Uchimura [25] proved that

$$\sum_{k=1}^{n}(-1)^{k-1}\begin{bmatrix} n \\ k \end{bmatrix}_q \frac{q^{\binom{k+1}{2}}}{1-q^{k+m}} = \sum_{k=1}^{n}\frac{q^k}{1-q^k}\begin{bmatrix} k+m \\ m \end{bmatrix}_q^{-1}, \quad m \geqslant 0. \tag{1.4}$$

Dilcher [9] established the following multiple series generalization of (4.8):

$$\sum_{k=1}^{n}(-1)^{k-1}\begin{bmatrix} n \\ k \end{bmatrix}_q \frac{q^{\binom{k}{2}+km}}{(1-q^k)^m} = \sum_{1\leqslant k_1\leqslant\cdots\leqslant k_m\leqslant n}\frac{q^{k_1+\cdots+k_m}}{(1-q^{k_1})\cdots(1-q^{k_m})}. \tag{1.5}$$

Prodinger [21] proved that

$$\sum_{\substack{k=0\\k\neq m}}^{n}(-1)^{k-1}\begin{bmatrix} n \\ k \end{bmatrix}_q \frac{q^{\binom{k+1}{2}}}{1-q^{k-m}} = (-1)^m q^{\binom{m+1}{2}}\begin{bmatrix} n \\ m \end{bmatrix}_q \sum_{\substack{k=0\\k\neq m}}^{n}\frac{q^{k-m}}{1-q^{k-m}}, \quad 0 \leqslant m \leqslant n. \tag{1.6}$$

Using partial fraction decomposition the second author [28, (7)] obtained the following common generalization of Dilcher's identity (1.5) and of some identities due to Fu and Lascoux [11,12]:

$$\sum_{k=i}^{n}(-1)^{k-i}\begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} k \\ i \end{bmatrix}_q \frac{q^{\binom{k-i}{2}+km}}{(1-zq^k)^m} = \frac{q^i(q;q)_{i-1}(q;q)_n}{(q;q)_i(zq;q)_n} h_{m-1}\left(\frac{q^i}{1-zq^i},\ldots,\frac{q^n}{1-zq^n}\right), \tag{1.7}$$

where $1 \leqslant i \leqslant n$ and $h_k(x_1,\ldots,x_n)$ is the k-th homogeneous symmetric polynomial in $x_1,\ldots,x_n$ defined by

$$h_k(x_1,\ldots,x_n) = \sum_{1\leqslant i_1\leqslant\cdots\leqslant i_k\leqslant n} x_{i_1}\cdots x_{i_k} = \sum_{\alpha_1+\cdots+\alpha_n=k} x_1^{\alpha_1}\cdots x_n^{\alpha_n}.$$

We note that Ismail and Stanton [18, Theorem 2.2] have rediscovered the $i=1$ case of (1.7) as well as some other results in [28].

In this paper we shall give a different kind of generalizations of (1.4)–(1.6). Our starting point is an identity of Chen and Fu [8, (3.3)], which corresponds to the $(r,x)=(0,1)$ case of the following result.

Theorem 1.1. *Let $m,n \geqslant 0$ and $0 \leqslant r \leqslant m$. Then*

$$\sum_{k=0}^{m}\frac{(-1)^k p^{\binom{k+1}{2}-rk}}{(p;p)_k(p;p)_{m-k}(xp^k;q)_{n+1}} = x^r\sum_{k=0}^{n}\frac{(-1)^k q^{\binom{k+1}{2}+rk}}{(q;q)_k(q;q)_{n-k}(xq^k;p)_{m+1}}. \tag{1.8}$$