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A unified approach for the Hankel determinants of classical combinatorial numbers

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ABSTRACT

We give a general formula for the determinants of a class of Hankel matrices which arise in combinatorics theory. We revisit and extend existent results on Hankel determinants involving the sum of consecutive Catalan, Motzkin and Schröder numbers and we prove a conjecture of [10] about the recurrence relations satisfied by the Hankel transform of linear combinations of Catalans numbers.

1. Introduction

Let $a = \{a_n\}_{n\in\mathbb{N}}$ denote a sequence of numbers. The $n \times n$ matrix

$$\mathcal{H}_n(a) = (a_{i+j})_{0\le i,j\le n-1},$$

is called Hankel matrix. If $h_n = \det(\mathcal{H}_n(a))$, then the sequence $\{h_n\}_{n\ge 0}$ is referenced as the Hankel transform of the sequence a and was widely investigated in numerous papers. Hankel determinants are particularly interesting when applied to classical combinatorial sequences arising from the lattice path enumerations and have attracted an increasing amount of attention recently [1,2,5,7,23,24]. One of the most popular themes in this context is to consider the determinant of the Hankel matrix generated by the sequence that is linear combinations of the sequences $\{a_n\}$ where $a_n = C_n$, M_n and R_n are Catalan, Motzkin or Schröder numbers respectively. For instance, Hankel determinant evaluations such as

$$\det\left((C_{i+j})_{0\le i,j\le n-1}\right) = 1,$$
$$\det\left((M_{i+j})_{0\le i,j\le n-1}\right) = 1,$$
$$\det\left((R_{i+j})_{0\le i,j\le n-1}\right) = 2^{\binom{n}{2}},$$

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or these involving consecutive terms have been addressed numerous times in the literature. Among the method employed to prove such formulae we cite the combinatorial methods based on the Lindström–Gessel–Viennot lemma on non-intersecting lattice paths and orthogonal polynomials. The reader is referred to Krattenthaler papers [16,17].

In this paper, our main focus is an overall generalization of these results. We evaluate $\det(\mathcal{H}_n(b))$ for $b = \{b_n\}$ of the form

$$b_n = \sum_{k=0}^{r} \lambda_k a_{n+k},$$

where $\lambda_0, \lambda_1, \ldots, \lambda_{r-1}, \lambda_r$, $r \geq 1$, are complex numbers such that $\lambda_r = 1$. We shall assume that $\det\left((a_{i+j})_{0\leq i,j\leq n-1}\right) \neq 0$ for all $n \geq 0$ and we denote by $\mathcal{L}$ the linear functional on the vector space of all polynomials defined by

$$\mathcal{L}(x^n) = a_n \quad \text{for } n = 0, 1, \ldots$$

With $\{a_n\}$ we associate the monic orthogonal polynomial sequence $\{p_n(x)\}_{n\in\mathbb{N}}$ [6] such that p_n is monic of degree n and

$$\mathcal{L}(p_n p_m) = 0 \quad \text{for } n \neq m.$$

We remark that $b_n = \sum_{k=0}^{r} \lambda_k a_{n+k} = \mathcal{L}(x^n q)$, where

$$q(x) = x^r + \lambda_{r-1}x^{r-1} + \ldots + \lambda_0.$$

The r-kernel $\mathcal{K}_{n,P}^{(r)}$ of $P = \{p_n\}_{n\in\mathbb{N}}$ is defined by

$$\mathcal{K}_{n,P}^{(r)}(x_1, x_2, \ldots, x_r) = \frac{\det\left((p_{n+i-1}(x_j))_{1\leq i,j\leq r}\right)}{\prod_{1\leq i<j\leq r}(x_j - x_i)}$$

for $r \geq 2$ and $\mathcal{K}_{n,P}^{(1)}(x) = p_n(x)$. As it will be shown latter, $\mathcal{K}_{n,P}^{(r)}(x_1, x_2, \ldots, x_r)$ is a polynomial of the variables $x_1, x_2, \ldots$ and x_r.

The following theorem constitutes our main result:

Theorem 1. *We have*

$$\det(\mathcal{H}_n(b)) = (-1)^{nr} \det(\mathcal{H}_n(a))\, \mathcal{K}_{n,P}^{(r)}(\alpha_1, \alpha_2, \ldots, \alpha_r), \tag{1.1}$$

where $\alpha_1, \alpha_2, \ldots, \alpha_r$ are the zeros of q.

In most examples considered in the existing literature, b_n has a specific pattern. Namely

$$b_n = a_{n+r} - c a_{n+r-1}, \text{ with } c \in \mathbb{C}.$$

Theorem 2. *We have for $c \neq 0$:*