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Symmetry and nonexistence of positive solutions of integral systems with Hardy terms [☆]

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ABSTRACT

Let α, s and t be real numbers satisfying $0 < \alpha < n$ and $0 \le s, t < \alpha$, we consider the following weighted system of partial differential equations

$$\begin{cases} (-\Delta)^{\frac{\alpha}{2}} u(x) = |x|^{-s} v^q, \\ (-\Delta)^{\frac{\alpha}{2}} v(x) = |x|^{-t} u^p, \end{cases}$$

where $p, q > 1$. We first establish the equivalence between partial differential system and weighted integral system

$$\begin{cases} u(x) = \int\limits_{R^n} \frac{v^q(y)}{|x-y|^{n-\alpha}|y|^s} dy, \\ v(x) = \int\limits_{R^n} \frac{u^p(y)}{|x-y|^{n-\alpha}|y|^t} dy. \end{cases}$$

Then, in the critical case of $\frac{n-s}{q+1} + \frac{n-t}{p+1} = n - \alpha$, we show that every pair of positive solutions $(u(x), v(x))$ is radially symmetric about the origin. While in the subcritical case, we prove the nonexistence of positive solutions.

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1. Introduction

We consider the following weighted partial differential system in R^n

$$\begin{cases} (-\Delta)^{\frac{\alpha}{2}} u(x) = |x|^{-s} v^q, \\ (-\Delta)^{\frac{\alpha}{2}} v(x) = |x|^{-t} u^p, \end{cases} \tag{1.1}$$

where $0 < \alpha < n$ and $0 \leq s, t < \alpha$. We say that (1.1) is in critical case when p, q satisfy

$$\frac{n-s}{q+1} + \frac{n-t}{p+1} = n - \alpha; \tag{1.2}$$

it is in super critical case when "<" holds; and in subcritical case when ">" holds, that is when

$$\frac{n-s}{q+1} + \frac{n-t}{p+1} > n - \alpha.$$

The case $\alpha = 2$ but $s \neq 0$ and/or $t \neq 0$ has been studied by de Figueiredo [8] and Liu and Yang [17]. Both the papers considered the Dirichlet problem of (1.1) in a bounded smooth domain. In this special case, Jin [11] proved that weak solutions of system (1.1) are radially symmetric.

In the case $s = t = 0$ and $\alpha = 2$, (1.1) has been studied by many authors. It is known that if (p, q) lies on or above the Sobolev hyperbola (in critical or super critical case), then system (1.1) admits some positive classical solutions in the whole space R^n (see [7,19]). The well known Lane–Emden conjecture states that, in the subcritical case, system (1.1) possesses no positive classical solutions.

Recently, when $s = t = 0$ and α is a real number between 0 and n, (1.1) has been investigated by Chen and Li [3].

In this paper, we prove that, under some integrability conditions, the positive solutions of (1.1) are radially symmetric in critical case; and when p and q are both subcritical, that is $p < 2*(t) - 1 := \frac{n+\alpha-2t}{n-\alpha}$ and $q < 2*(s) - 1 := \frac{n+\alpha-2s}{n-\alpha}$, system (1.1) possesses no positive solutions. Actually, we consider the corresponding weighted integral system in R^n:

$$\begin{cases} u(x) = \displaystyle\int_{R^n} \frac{v^q(y)}{|x-y|^{n-\alpha}|y|^s} dy, \\ v(x) = \displaystyle\int_{R^n} \frac{u^p(y)}{|x-y|^{n-\alpha}|y|^t} dy. \end{cases} \tag{1.3}$$

We first prove that system (1.1) is equivalent to the integral system (1.3). Our result is

Theorem 1. *Let $\alpha = 2m$ be an even number less than n. Assume that $(u(x), v(x))$ is a pair of positive solutions of (1.1), then a constant multiple of $(u(x), v(x))$ satisfies (1.3).*

For other real values of α, we define the solution of (1.1) in the distribution sense, that is $u, v \in H^{\frac{\alpha}{2}}(R^n)$ and satisfy

$$\int_{R^n} (-\Delta)^{\frac{\alpha}{4}} u (-\Delta)^{\frac{\alpha}{4}} \phi dx = \int_{R^n} |x|^{-s} v^q(x) \phi(x) dx, \tag{1.4}$$

$$\int_{R^n} (-\Delta)^{\frac{\alpha}{4}} v (-\Delta)^{\frac{\alpha}{4}} \phi dx = \int_{R^n} |x|^{-t} u^p(x) \phi(x) dx, \tag{1.5}$$