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# Rearranging series of vectors on a small set

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ABSTRACT

We consider a variation of the Lévy–Steinitz theorem where one rearranges only a small number of terms. In particular, we study the set of obtainable rearrangements of a multidimensional series by a permutation $\sigma : \omega \to \omega$ such that $\{n \in \omega : \sigma(n) \neq n\} \in \mathcal{I}$ for some proper ideal $\mathcal{I} \subset \mathcal{P}(\omega)$.



## 1. Introduction

The classic theorem of Riemann [5] states that for any conditionally convergent series of reals $\sum_{n\in\omega} a_n$ and any real number $a$ there exists a permutation of terms $\sigma$ such that $\sum_{n\in\omega} a_{\sigma(n)} = a$.

In [8] Wilczyński strengthened the result, adding that we can take the permutation such that $\{n \in \omega : \sigma(n) \neq n\} \in \mathcal{I}_d$, where $\mathcal{I}_d$ denotes the ideal of sets of asymptotic density zero. One might say that this means that the permutation changes only a small number of terms. In the same paper Wilczyński asked what other ideals can be put in the place of $\mathcal{I}_d$. This question was answered by Filipów and Szuca in [1], where they proved that those ideals are exactly the ones which cannot be extended to a summable ideal.

Riemann's theorem has been generalized to a multidimensional case by Lévy [4] and Steinitz [7].

In this paper we consider an approach similar to the one-dimensional case, i.e. we study the sets of those points in a multidimensional Euclidean space which are obtained by rearranging a small amount of the series' terms. While the original Lévy–Steinitz theorem works for $\mathbb{R}^m$, where $m$ is any positive natural, we decided to focus solely on the case $m = 2$ (i.e. the plane). We divided this into two parts – in Section 3 we work on a situation where the set of obtainable sums is a line on the plane, while in Section 4 it is the entire plane (which turned out to be a much more delicate counterpart).

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## 2. Preliminaries

By $\omega$ we denote the set of natural numbers, and by $\mathcal{P}(\omega)$ its powerset. An ideal $\mathcal{I} \subset \mathcal{P}(\omega)$ is a family which is closed under taking subsets and finite unions. If $\omega \notin \mathcal{I}$ then an ideal is proper. In this paper we will assume that all ideals are proper.

We will use the following notation: $a^+ = \max\{a, 0\}$, $a^- = \min\{a, 0\}$.

**Definition 2.1.** Let $(v_n)_{n\in\omega}$ be a sequence in $\mathbb{R}^m$, $m \in \{1, 2, 3, ...\}$.

$$S\left(\sum_{n\in\omega} v_n\right) = \left\{v \in \mathbb{R}^m : \exists_{\sigma:\omega\to\omega}\ \sigma - \text{permutation},\ \sum_{n\in\omega} v_{\sigma(n)} = v\right\}.$$

Using this notation we can formulate Riemann's derangement theorem in the following way.

**Theorem 2.2.** *(See [5].) Let $\sum_{n\in\omega} a_n$ be a conditionally convergent series of reals. Then*

$$S\left(\sum_{n\in\omega} a_n\right) = \mathbb{R}.$$

By *supp*$(\sigma)$ we'll denote the support of the permutation $\sigma$, that is the set $\{n \in \omega : \sigma(n) \neq n\}$.

**Definition 2.3.** Let $(v_n)_{n\in\omega}$ be a sequence in $\mathbb{R}^m$, $m \in \{1, 2, 3, ...\}$. Also let $\mathcal{I} \subset \mathcal{P}(\omega)$ be an ideal.

$$S_{\mathcal{I}}\left(\sum_{n\in\omega} v_n\right) = \left\{v \in \mathbb{R}^m : \exists_{\sigma:\omega\to\omega}\ \sigma - \text{permutation},\ \sum_{n\in\omega} v_{\sigma(n)} = v, \mathit{supp}(\sigma) \in \mathcal{I}\right\}.$$

Recall the definition of the ideal of sets of asymptotic density zero.

$$\mathcal{I}_d = \left\{A \subset \omega : \limsup_{n\to\infty} \frac{|\{0, 1, \ldots, n-1\} \cap A|}{n} = 0\right\}.$$

**Theorem 2.4.** *(See [8].) Let $\sum_{n\in\omega} a_n$ be a conditionally convergent series of reals. Then*

$$S_{\mathcal{I}_d}\left(\sum_{n\in\omega} a_n\right) = \mathbb{R}.$$

We will say that an ideal $\mathcal{I} \subset \mathcal{P}(\omega)$ has the $(W)$ property if for any conditionally convergent series of reals $\sum_{n\in\omega} a_n$ there exists $W \in \mathcal{I}$ such that $\sum_{n\in W} a_n$ is also conditionally convergent. Additionally, we will say that an ideal $\mathcal{I} \subset \mathcal{P}(\omega)$ has the $(R)$ property if for any conditionally convergent series of reals $\sum_{n\in\omega} a_n$ and any $a \in \mathbb{R}$ there exists a permutation $\sigma : \omega \to \omega$ such that $\sum_{n\in\omega} a_{\sigma(n)} = a$ and *supp*$(\sigma) \in \mathcal{I}$.

Hence we can express Theorem 2.4 by saying that $\mathcal{I}_d$ has the $(R)$ property. In [8] Wilczyński asked to find a full characterization of ideals fulfilling this property. This question was answered by Filipów and Szuca [1].

First, recall the definition of the summable ideals. Let $(a_n)_{n\in\omega}$ be such sequence of reals that $a_n \geqslant 0$, $n \in \omega$, $a_n \to 0$ and $\sum_{n\in\omega} a_n = +\infty$. The family

$$\mathcal{I}_{a_n} = \left\{A \subset \omega : \sum_{n\in A} a_n < \infty\right\}$$

is an ideal which we call the summable ideal (determined by the sequence $a_n$).