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Relationship between the concave integrals and the pan-integrals on finite spaces

Yao Ouyang [a,*], Jun Li [b], Radko Mesiar [c,d]

[a] Faculty of Science, Huzhou University, Huzhou, Zhejiang 313000, China
[b] School of Sciences, Communication University of China, Beijing 100024, China
[c] Faculty of Civil Engineering, Slovak University of Technology, Radlinského 11, 813 68 Bratislava, Slovakia
[d] UTIA CAS, Pod Vodárenskou věží 4, 182 08 Prague, Czech Republic

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ABSTRACT

This study discusses the relationship between the concave integrals and the pan-integrals on finite spaces. The *minimal atom* of a monotone measure is introduced and some properties are investigated. By means of two important structure characteristics related to minimal atoms: *minimal atoms disjointness property* and *subadditivity for minimal atoms*, a necessary and sufficient condition is given that the concave integral coincides with the pan-integral with respect to the standard arithmetic operations $+$ and $\cdot$ on finite spaces. Following this result, we have shown that these two integrals coincide if the underlying monotone measure is subadditive.

1. Introduction

In non-additive measure and integral theory, several prominent nonlinear integrals with respect to monotone measure (or capacity) have been defined and discussed in detail. Among them, we mention the Choquet integral [2] (see also [3,19]), the pan-integral introduced by Yang [27] (see also [25,24]) and the concave integral introduced by Lehrer [9] (see also [10]). Although all the three types of integrals coincide with the Lebesgue integral in the case where the monotone measure is σ-additive, they are significantly different from each other. The Choquet integral is based on chains of sets, while, similarly to the Lebesgue integral, the pan-integral deals with disjoint finite set systems. Finally, the concave integral deals with arbitrary finite set systems, see [16]. Note that all these integrals can be seen as particular instances of decomposition integrals recently introduced by Even and Lehrer [5], see also [16].

* Corresponding author.
E-mail addresses: oyy@hutc.zj.cn (Y. Ouyang), lijun@cuc.edu.cn (J. Li), mesiar@math.sk (R. Mesiar).

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Lehrer [9] showed that the Choquet integral is less than or equal to the concave integral and these two integrals coincide if and only if the underlying capacity ν is convex (also known as supermodular, see [3,19]). In [24] it was proved that the pan-integral with respect to the usual addition $+$ and usual multiplication $\cdot$ is less than or equal to the Choquet integral if the underlying monotone measure μ is superadditive, while the pan-integral is greater than or equal to the Choquet integral if μ is subadditive.

This paper will focus on the relationship between the concave integrals and the pan-integrals. We explore the conditions under which the concave integral coincides with the pan-integral w.r.t. the usual addition $+$ and usual multiplication $\cdot$.

We shall introduce the concept of *minimal atom* of a monotone measure and investigate its properties. As a special kind of atom of monotone measure, the minimal atom plays an essential role in our discussions. By means of minimal atoms we describe the relationship between the concave integrals and pan-integrals on finite spaces. We introduce two important concepts related to minimal atoms: *minimal atoms disjointness property* which is weaker than weak null-additivity, and *subadditivity w.r.t. minimal atoms* which is weaker than subadditivity. Our main results are in Section 4. We shall show that on finite spaces the concave integral coincides with the pan-integral w.r.t. the usual addition $+$ and usual multiplication $\cdot$ if and only if the underlying monotone measure μ satisfies both the minimal atoms disjointness property and subadditivity w.r.t. minimal atoms. As a direct corollary, we obtain that on finite spaces the above mentioned two integrals coincide if the monotone measure μ is subadditive.

2. Preliminaries

Let X be a nonempty set and $\mathcal{F}$ a σ-algebra of subsets of X. $\mathbf{F}$ denotes the class of all finite nonnegative real-valued measurable functions on $(X, \mathcal{F})$. Unless stated otherwise all the subsets mentioned are supposed to belong to $\mathcal{F}$, and all the functions mentioned are supposed to belong to $\mathbf{F}$.

We assume that μ is a monotone measure on $(X, \mathcal{F})$, i.e., $\mu : \mathcal{F} \to [0, +\infty]$ is an extended real-valued set function satisfying the following conditions:

(1) $\mu(\emptyset) = 0$ and $\mu(X) > 0$;
(2) $\mu(A) \le \mu(B)$ whenever $A \subset B$ and $A, B \in \mathcal{F}$. (monotonicity)

When μ is a monotone measure, the triple $(X, \mathcal{F}, \mu)$ is called a monotone measure space [19,24]. In some literature, the monotone measure constraint by $\mu(X) = 1$ is also known as capacity or fuzzy measure, or nonadditive probability, etc. (see [3,9,17,23,25]).

A monotone measure μ is said to be *weakly null-additive* [24], if $\mu(A \cup B) = 0$ whenever $\mu(A) = \mu(B) = 0$; *subadditive* if $\mu(A \cup B) \le \mu(A) + \mu(B)$ for any $A, B \in \mathcal{F}$.

The concept of a pan-integral [24,27] involves two binary operations, the pan-addition $\oplus$ and pan-multiplication $\otimes$ of real numbers (see also [15,19,20,25,24]). To be able to compare the concave and pan-integrals, in this paper we consider the pan-integrals based on the standard addition $+$ and multiplication $\cdot$ only. We recall the following definition.

Definition 2.1. Let $(X, \mathcal{F}, \mu)$ be a monotone measure space and $f \in \mathbf{F}$. The pan-integral of f on X w.r.t. the usual addition $+$ and usual multiplication $\cdot$ is given by

$$\int^{pan} f d\mu = \sup\left\{\sum_{i=1}^{n} \lambda_i \mu(A_i) : \sum_{i=1}^{n} \lambda_i \chi_{A_i} \le f,\ \{A_i\}_{i=1}^{n} \subset \mathcal{F} \text{ is a partition of } X, \lambda_i \ge 0, n \in \mathbb{N}\right\}.$$