J. Math. Anal. Appl. 424 (2015) 1143–1160

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

A positive and bounded finite element approximation of the generalized Burgers–Huxley equation

V.J. Ervin [a,*], J.E. Macías-Díaz [b], J. Ruiz-Ramírez [a]

[a] *Department of Mathematical Sciences, Clemson University, Clemson, SC 29634, USA*
[b] *Departamento de Matemáticas y Física, Universidad Autónoma de Aguascalientes, Aguascalientes, Ags. 20131, Mexico*

ARTICLE INFO

Article history:
Received 17 October 2014
Available online 28 November 2014
Submitted by W. Layton

Keywords:
Finite element method
Convection–diffusion–reaction
Nonnegativity
Boundedness

ABSTRACT

We present a finite element scheme capable of preserving the nonnegative and bounded solutions of the generalized Burgers–Huxley equation. Proofs of existence and uniqueness of a solution to the continuous problem together with some results concerning the boundedness and the nonnegativity of the solution are given. Under appropriate conditions on the mesh and the initial and boundary data, boundedness and nonnegativity of the finite element approximation are established. An *a priori* error estimate for the approximation is also derived. Numerical experiments are presented which support the derived theoretical results.

1. Introduction

The class of advection–diffusion–reaction problems has been used extensively to model different physical processes such as atmospheric air quality [20], the mobility of fish populations [27], nuclear waste disposal [14], pattern formation [23] and the reactive transport of contaminants [16]. An important member of this class and the object of study in this paper is the generalized Burgers–Huxley equation, which has found application in biology [28], electrodynamics [29] and transport phenomena [1]. For $\Omega \subset \mathbb{R}^2$, the nonlinear partial differential equation under consideration is:

$$\frac{\partial u}{\partial t} + \alpha u^p \left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \right) - \Delta u - u g(u) = 0, \quad (\mathbf{x}, t) \in \Omega \times (0, T), \tag{1}$$

where $\alpha \in \mathbb{R}^+$ is the advection coefficient, $\gamma \in (0, 1)$, $p \geq 1$ are parameters and

$$g(u) = \beta \big(1 - u^p\big)\big(u^p - \gamma\big), \quad \beta > 0. \tag{2}$$

* Corresponding author.
E-mail addresses: vjervin@clemson.edu (V.J. Ervin), jemacias@correo.uaa.mx (J.E. Macías-Díaz), javier@clemson.edu (J. Ruiz-Ramírez).

http://dx.doi.org/10.1016/j.jmaa.2014.11.047

The numerical approximation in 1D of (1) using a variety of techniques has received considerable attention. One approach is the Adomian decomposition [15] which needs no discretization, linearization or perturbation, and provides an analytical solution in the form of a power series using Adomian polynomials. A second approach, which can be classified as a Lagrange multiplier method, is the variational iteration method (VIM) [2]. This technique uses a linearizion assumption as an initial approximation and later, through a correction functional, the approximation is made more precise. Results have shown that in some cases one iteration of VIM is of comparable accuracy to a 5-term Adomian solution. A third approach which belongs to the class of interpolation techniques is the spectral collocation method (SC) [7]. Using Chebyshev–Gauss–Lobatto collocation points, an interpolant polynomial is constructed and a differential operator in terms of the grid point values is computed. The matrices that appear in SC are generally ill-conditioned and some preconditioning is required. A fourth approach is the use of Haar wavelets [4]. The procedure relies on the decomposition of an $L^2([a,b])$ function in terms of the orthogonal Haar basis. The matrices that arise in the computations are sparse and the accuracy is in general high even with few collocation points. A fifth approach is finite differences [26], where a Runge–Kutta method of order 4 is used in time and coupled with a 10-th order finite difference scheme in space. A final example of approximation methods investigated in 1D is the finite element method [19], where a three-step Taylor–Galerkin finite element scheme is implemented to approximate a problem with the diffusion term being multiplied by a perturbation parameter ε.

Of interest is the design of numerical methods capable of preserving the nonnegative and bounded solutions of (1). Among all the aforementioned methods, only finite differences and finite element methods have been successful in being able to supply proofs for the conservation of those properties. Works in this direction in the case of finite differences include [21,22,25]. If we restrict ourselves to the smaller class of linear parabolic problems of the form $u_t - \nabla \cdot (\kappa \nabla u) = f$, where κ satisfies $0 < \kappa_{\min} \leq \kappa(\mathbf{x}) \leq \kappa_{\max}$ and f is bounded, for finite differences and finite element the (discrete) nonnegativity preservation property is equivalent to the (discrete) maximum principle, a bridge that was established in [10]. Some articles on finite elements concerning maximum principles for advection–reaction–diffusion problems are [6,9,11,18].

In this document we propose a finite element scheme capable of preserving the nonnegative and bounded solutions of (1) under suitable conditions for the computational parameters Δt, h, and on the triangulation $\mathcal{T}_h$ of the region Ω. In Section 2 we establish a result concerning the existence and uniqueness of a solution to the continuous problem (1). Moreover, we prove that for certain initial and boundary conditions, any classical solution to (1) satisfies certain boundedness conditions. Then, in Section 3, we state the weak formulation and prove the boundedness of the solution in this setting. The discrete weak formulation is introduced in Section 4 and therein we show the computability of the scheme and state parallel results to the ones derived for the continuous problem. In Section 5 we provide an *a priori* error estimate which guarantees that the method is of the first order accurate, with respect to the space and time discretizations. Finally, in Section 6, numerical experiments are presented which support our theoretical results.

2. Continuous problem

In this section we establish the existence of a solution to (1) for some time interval of positive length $[0, T_0]$. Then, we show that under suitable boundary and initial conditions, the solution u is nonnegative and bounded.

Before we state our first result, we introduce the following definition.

Definition 1. The real valued function $F(\mathbf{x}, t, z_1, \dots, z_r)$, $F : \Omega \times \mathbb{R}^+ \cup \{0\} \times \mathbb{R}^r \to \mathbb{R}$, with $\Omega \subset \mathbb{R}^2$, is locally Hölder continuous with respect to $(\mathbf{x}, t)$ if for all $B \subset \overline{\Omega} \times [0, T]$, B compact, there exist a constant $C > 0$ and $0 < \kappa < 1$ such that