J. Math. Anal. Appl. 424 (2015) 1161–1185

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Sup–Inf explicit formulas for minimal Lipschitz extensions for 1-fields on $\mathbb{R}^n$

Erwan Y. Le Gruyer *, Thanh Viet Phan

INSA de Rennes & IRMAR, 20, Avenue des Buttes de Coësmes, CS 70839, F-35708 Rennes Cedex 7, France

A R T I C L E   I N F O

Article history:
Received 1 July 2014
Available online 3 December 2014
Submitted by H. Frankowska

Keywords:
Minimal
Lipschitz
Extension
Differentiable function
Convex analysis

A B S T R A C T

We study the relationship between the Lipschitz constant of 1-field introduced in [12] and the Lipschitz constant of the gradient canonically associated with this 1-field. Moreover, we produce two explicit formulas which are two extremal minimal Lipschitz extensions for 1-fields. As a consequence of the previous results, for the problem of minimal extension by Lipschitz continuous functions from $\mathbb{R}^m$ to $\mathbb{R}^n$, we produce explicit formulas similar to those of Bauschke and Wang (see [7]). Finally, we show that Wells's extensions (see [24]) of 1-fields are absolutely minimal Lipschitz extensions when the domain of 1-field to expand is finite. We provide a counter-example showing that this result is false in general.

1. Introduction

Let Ω be a subset of Euclidean space $\mathbb{R}^n$. We suppose that Ω has at least two elements. Let $\mathscr{P}^1(\mathbb{R}^n,\mathbb{R})$ be the set of first degree polynomials mapping $\mathbb{R}^n$ to $\mathbb{R}$, i.e.

$$\mathscr{P}^1\big(\mathbb{R}^n,\mathbb{R}\big) \triangleq \big\{P : a \in \mathbb{R}^n \mapsto P(a) = p + \langle v, a\rangle,\ p \in \mathbb{R},\ v \in \mathbb{R}^n\big\}.$$

Let us consider a 1-field F on domain $\operatorname{dom}(F) \triangleq \Omega$ defined by

$$\begin{aligned} F : \Omega &\to \mathscr{P}^1\big(\mathbb{R}^n,\mathbb{R}\big) \\ x &\mapsto F(x)(a) \triangleq f_x + \langle D_x f; a - x\rangle, \end{aligned} \tag{1}$$

where $a \in \mathbb{R}^n$ is the evaluation variable of the polynomial $F(x)$ and $f : x \in \Omega \mapsto f_x \in \mathbb{R}$, $Df : x \in \Omega \mapsto D_x f \in \mathbb{R}^n$ are mappings associated with F. We will always use capital letters to denote the 1-field and small letters to denote these mappings.

* Corresponding author.
E-mail addresses: Erwan.Le-Gruyer@insa-rennes.fr (E.Y. Le Gruyer), Thanh-Viet.Phan@insa-rennes.fr (T.V. Phan).

http://dx.doi.org/10.1016/j.jmaa.2014.11.067
0022-247X/

The Lipschitz constant of F introduced in [12] is

$$\Gamma^1(F;\Omega) \triangleq \sup_{\substack{x,y\in\Omega\\ x\neq y}} \Gamma^1(F;x,y), \tag{2}$$

where

$$\Gamma^1(F;x,y) \triangleq 2\sup_{a\in\mathbb{R}^n} \frac{|F(x)(a)-F(y)(a)|}{\|x-a\|^2+\|y-a\|^2}. \tag{3}$$

If $\Gamma^1(F;\Omega) < +\infty$, then the Whitney conditions [26], or Glaeser in [10] are satisfied and the 1-field F can be extended on $\mathbb{R}^n$: there exists $g \in \mathscr{C}^{1,1}(\mathbb{R}^n,\mathbb{R})$ such that $g(x) = f_x$ and $\nabla g(x) = D_x f$ for all $x \in \Omega$ where ∇g is the usual gradient. Moreover, from [12, Theorem 2.6] we can find g which satisfies

$$\Gamma^1\big(G;\mathbb{R}^n\big) = \Gamma^1(F;\Omega),$$

where G is the 1-field associated to g, i.e.

$$G(x)(y) = g(x) + \big\langle \nabla g(x), y-x\big\rangle, \quad x\in\Omega,\ y\in\mathbb{R}^n.$$

It means that the Lipschitz constant does not increase when extending F by G. We say that G is a minimal Lipschitz extension (MLE for short) of F and we have

$$\Gamma^1\big(G;\mathbb{R}^n\big) = \inf\big\{\mathrm{Lip}\big(\nabla h;\mathbb{R}^n\big) : h(x)=f_x,\ \nabla h(x) = D_x f,\ x\in\Omega,\ h\in\mathscr{C}^{1,1}\big(\mathbb{R}^n,\mathbb{R}\big)\big\},$$

where the notation $\mathrm{Lip}(u;.)$ means that

$$\mathrm{Lip}(u;x,y) \triangleq \frac{\|u(x)-u(y)\|}{\|x-y\|}, \quad x,y\in\Omega,\ x\neq y, \quad \text{and} \quad \mathrm{Lip}(u;\Omega) \triangleq \sup_{x\neq y\in\Omega} \mathrm{Lip}(u;x,y). \tag{4}$$

It is worth asking what is it the relationship between $\Gamma^1(F;\Omega)$ and $\mathrm{Lip}(Df;\Omega)$? From [12], we know that $\mathrm{Lip}(Df;\Omega) \le \Gamma^1(F;\Omega)$. In the special case $\Omega = \mathbb{R}^n$ we have $\mathrm{Lip}(Df;\mathbb{R}^n) = \Gamma^1(F;\mathbb{R}^n)$ but in general the formula $\mathrm{Lip}(Df;\Omega) = \Gamma^1(F;\Omega)$ is untrue. In this paper we will prove that if Ω is an open subset of $\mathbb{R}^n$ then

$$\Gamma^1(F;\Omega) = \max\big\{\Gamma^1(F;\partial\Omega), \mathrm{Lip}(Df;\Omega)\big\}, \tag{5}$$

where $\partial\Omega$ is a boundary of Ω. Moreover, if Ω is a convex subset of $\mathbb{R}^n$ then

$$\Gamma^1(F;\Omega) \le 2\,\mathrm{Lip}(Df;\Omega). \tag{6}$$

To make the connection between $\Gamma^1(F;\Omega)$ and $\mathrm{Lip}(Df;\Omega)$, it is important to know the set of uniqueness of minimal extensions of F when Ω has two elements (this study was performed in [14]). Indeed, many results of Section 3 use this knowledge. For further more details see Section 3.

In Section 4, we present two MLEs U^+ and U^- of F of the form

$$U^+ : x\in\mathbb{R}^n \mapsto U^+(x)(y) \triangleq u^+(x) + \big\langle D_x u^+; y-x\big\rangle, \quad y\in\mathbb{R}^n, \tag{7}$$

where

$$u^+(x) \triangleq \sup_{v\in\Lambda_x}\inf_{a\in\Omega} \Psi^+(F,x,a,v), \qquad D_x u^+ \triangleq \arg\sup_{v\in\Lambda_x}\inf_{a\in\Omega} \Psi^+(F,x,a,v), \tag{8}$$