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Numerical representability of preferences by economic functions

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ABSTRACT

This article is concerned with the representation of the individual's preference relation by numerical functions, which are well-known in economics. The functions to be considered in this paper are the income compensation function, an appropriate distance function, and a Luenberger-type benefit function, which represent the individual's preferences under slightly different conditions. Since these functions have an appealing economic meaning, the individual's preferences can be represented in an appropriate economic context. Therefore, the preferences will not only be represented by an abstract unknown utility function, but by a function which can be constructed in the corresponding economic model. By this approach the hedonistic and problematic notion of utility can be avoided.

1. Introduction

This article studies the problem of representing preference relations by functions which have an economic meaning. These functions are essential in appropriate economic models like the theory of demand or welfare theory. In his famous book "Theory of Value" [9] G. Debreu has shown that every transitive, complete[2] and continuous relation $\succeq$ on a connected subset X of $\mathbb{R}^n$ can be represented by a continuous utility function $u : X \to \mathbb{R}$ such that for all $x, y \in X$:

$$x \succeq y \quad \Longleftrightarrow \quad u(x) \geqq u(y).$$

The proof uses profound mathematical knowledge and skills (Debreu [9, pp. 56–59]). Especially, we must be familiar with order-density. If the relation is not continuous, then in general, one has to require that there exists a countable subset of the given set X which is order dense in X (Debreu [9], Fishburn [12]). If the relation is monotone, then the difficulty of the proof reduces considerably (Mas-Colell, Whinston and

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[2] For all $x, y \in X : x \succeq y \vee y \succeq x$.

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Green [20, pp. 47–48]). In the present article we will reinvestigate the representation of a given preference relation by a function which has an appealing economic meaning, and which is principally constructive. Three different classes of functions will be considered: the income compensation function, the distance function, originally applied by R.W. Shephard to production theory (Shephard [27]), and a modified Luenberger benefit function (Luenberger [18]). Everybody, who is familiar with these functions, will realize that they are intuitive and useful representations of a given relation, when these functions are parts of appropriate economic models. The proofs of representability are quite simple. Only in the case of continuous representability of a given relation $\succeq$ by the income compensation function one needs a little more advanced mathematical tools.

Moreover, this article revisits the problem of the hedonistic and psychological meaning of utility in a scientific field like economics, which has been discussed for a long time in the economic literature (Samuelson [25], et al.). In his article "A note on the pure theory of consumer's behaviour" Samuelson developed an economic model "freed from any vestigial traces of the utility concept" (Samuelson [25, p. 71]). The aforementioned functions represent the agent's preferences without using the utility concept.

This article is organized as follows: We will start with the income compensation function as a representation of a given relation. In comparison, properties of $\succeq$ will be presented which imply that $\succeq$ can be represented by a distance function. Finally, a Luenberger-type benefit function as a representation of $\succeq$ will be studied. As we will see, the assumptions imposed on the underlying preferences will differ slightly.

2. The income compensation function

Income compensation functions are an important tool in the theory of consumer behavior. They were introduced to demand theory by Lionel McKenzie in 1957 [21]. By means of the income compensation functions, we can develop a model of consumer behavior based on the individual's preference relation and not on utility functions (Fuchs-Seliger [13]). We will come back to this point at the end of this section.

Income compensation functions can be applied in order to describe consumer's preferences by money-income. Therefore, the problematic and hedonistic notion of utility can be avoided in consumer theory, especially for the reason why the utility of goods is not cardinally measurable. In economics ordinal utility functions are usually assumed, meaning that the functions are determined up to a strictly increasing transformation. However, in this respect, utility is just an empty word. In order to avoid the problematic notion of utility, Paul Samuelson in 1938 developed a new description of consumer behavior based on demand functions. He introduces this approach by saying "I propose, therefore, that we start anew in direct attack upon the problem, dropping off the last vestiges of the utility analysis" (Samuelson [25, p. 62]). Paul Samuelson also suggested to measure "utility" by money-income (Samuelson [26, p. 1262]), and introduced the notion of money-metric utility functions (Samuelson [26]). Obviously, measuring the individual's "utility" by money-income has an intuitive meaning. Therefore, this paper will be especially concerned with income compensation functions.

Representability of a given relation $\succeq$ by an income compensation function has been investigated by several authors before. Especially, we refer to J. Weymark [29], S. Honkapohja [17], and J. Alcantud and A. Manrique [1]. While an income compensation function is in general defined for a complete, transitive, and continuous relation $\succeq$, the domain of $\succeq$ varies, and also the continuity of the income compensation function follows from different conditions. In this paper, we consider a closed set $X \subseteq \mathbb{R}^n_+$, $X \neq \emptyset$, which is interpreted as a set of commodity bundles, and the set $\mathbb{R}^n_{++}$ of strictly positive price vectors p. Firstly, it will only be assumed that $\succeq$ is reflexive. Then the income compensation (or minimum income) function can be defined as

$$m^0(p^0, x) := \inf_{y \in X} \{p^0 y \mid y \succeq x\}, \quad \text{for } p^0 \in \mathbb{R}^n_{++},\ x \in X.$$