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Asymptotic approximation for the solution to a semi-linear parabolic problem in a thick junction with the branched structure

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A B S T R A C T

We consider a semi-linear parabolic problem in a thick junction Ω_ε, which is the union of a domain Ω_0 and a lot of joined thin trees situated ε-periodically along some manifold on the boundary of Ω_0. The trees have finite number of branching levels. The following nonlinear Robin boundary condition $\partial_\nu v_\varepsilon + \varepsilon^{\alpha_i}\mu(t, x_2, v_\varepsilon) = \varepsilon^\beta g_\varepsilon$ is given on the boundaries of the branches from the i-th branching layer; $\{\alpha_i\}$ and β are real parameters. The asymptotic analysis of this problem is made as $\varepsilon \to 0$, i.e., when the number of the thin trees infinitely increases and their thickness vanishes. In particular, the corresponding homogenized problem is found and the existence and uniqueness of its solution in an anisotropic Sobolev space of multi-sheeted functions is proved. We construct the asymptotic approximation for the solution v_ε and prove the corresponding asymptotic estimate in the space $C([0,T]; L^2(\Omega_\varepsilon)) \cap L^2(0,T; H^1(\Omega_\varepsilon))$, which shows the influence of the parameters $\{\alpha_i\}$ and β on the asymptotic behavior of the solution.

1. Introduction

In recent years, materials with complex structure are widely used in engineering devices in many fields of science. It is known that some properties of materials are controlled by their geometrical structure. Therefore, the study of the influence of the material microstructure can improve its useful properties and reduce undesirable effects. Mathematical models for this study are boundary-value problems (BVPs) in domains with complex structures: perforated domains, grid-domains, domains with rapidly oscillating boundaries, thick junctions, etc.

In this paper, we begin to study asymptotic properties of solutions to BVPs in thick junctions of a new type, namely *thick junctions with the branched structure*. Such a thick junction is the union of some domain, which is called the junction's body, and a lot of joined thin trees situated ε-periodically along some manifold on the boundary of the junction's body. The trees have finite number of branching levels. The

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Fig. 1. Heat radiator shaped like a thick junction with the branched structure.

small parameter ε characterizes the distance between neighboring thin branches and also their thickness. As a first step, here we consider the case of a 2-D thick junction with the branched structure (see Fig. 2).

Various constructions of thick junction type are successfully used in nanotechnologies [12], microtechnique [13], modern engineering constructions (microstrip radiator, ferrite-filled rod radiator), as well as many physical and biological systems such as, for example, the structure of the intestine lining with different levels of absorption of nutrients on different part of the tissues. A number of new applications are envisioned, especially regarding efficient sensors (inertial, biological, chemical), signal processing filters (ultra large band), micro-fractal constructions: fractal antennas, fractal transistors, fractal heat radiators and so on. In Fig. 1 one can see a heat radiator with the branched structure that is used in order to receive or transmit more heat within a given total volume.

Such successful applications of thick-junction constructions have stimulated active learning BVPs in thick junctions with more complex structures: thick junctions with the thin junction's body [1–3], thick multi-level junctions [7,8,18], thick cascade junctions [4,16], where new qualitative results were obtained. Specifically, it was shown that processes in thick multi-level junctions behave as a "many-phase system" and thick cascade junctions have new kind of eigenvibrations. This means that materials with such micro-structures have some new properties.

It is often impossible to solve problems in thick junctions directly with numerical methods, because this would require too much CPU resources considering a large number of components of thick junctions (in some cases few thousands). Therefore development of new mathematical tools is necessary. One of them is asymptotic analysis of BVPs in thick junctions as $\varepsilon \to 0$, i.e., when the number of attached thin domains infinitely increases and their thickness decreases to zero. Asymptotic results give us the possibility to replace the original problem in a thick junction by the corresponding homogenized problem that is simpler and then apply computer simulation. In addition, in some cases it is possible to construct accurate and numerically implementable asymptotic approximations.

The rest of this paper is organized as follows.

The statement of the problem and features of the investigation are presented in Section 2.

In Section 3 we formally construct the leading terms of asymptotic expansions for the solution v_ε. The asymptotics consists of the outer expansions both in the junction's body and in each thin branch as well as the leading terms of inner expansions in a neighborhood both of the joint zone and of each branching level.

Then in Section 4, using the method of matched asymptotic expansions, we derive the corresponding nonstandard homogenized problem. The existence and uniqueness of its solution in an anisotropic Sobolev space of multi-sheeted functions is proved in Section 5.