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The Orlicz mean zonoid operator ☆

Lujun Guo [a,*], Gangsong Leng [b], Changmin Du [b]

[a] College of Mathematics and Information Science, Henan Normal University, Henan, 453007, PR China
[b] Department of Mathematics, Shanghai University, Shanghai, 200444, PR China

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ABSTRACT

In this paper, we introduce an Orlicz mean zonoid operator $Z_\phi : (\mathcal{K}^n)^2 \to \mathcal{K}^n$ which has Zhang's mean zonoid operator as a special case. We show that the new operator which actually maps $(\mathcal{K}^n)^2$ to $\mathcal{K}^n_0$ is commutative, $GL(n)$ covariant and continuous. We also establish an affine isoperimetric inequality for the Orlicz mean zonoid.

1. Introduction

A zonoid in $\mathbb{R}^n$ is an origin-symmetric convex body that can be approximated (in the Hausdorff metric) by finite Minkowski sums of line segments. It turns out that zonoids appear in many different contexts in convex geometry, physics, optimal control theory, and functional analysis (we refer the reader to [1–3,6,10,13,12,14,22,28,30–32]).

For a compact convex subset K, let $h(K,\cdot) = h_K(\cdot) : \mathbb{R}^n \to \mathbb{R}$ denote the support function of K; i.e., $h(K,x) = \max\{x \cdot y : y \in K\}$. An equivalent definition of zonoids is the following: A zonoid Z in $\mathbb{R}^n$ is a convex body whose support function at $x \in \mathbb{R}^n$ is given by

$$h_Z(x) = \frac{1}{2} \int\limits_{S^{n-1}} |x \cdot y| d\mu(y), \tag{1.1}$$

where μ is some positive, even Borel measure on S^{n-1} and $x \cdot y$ denotes the standard inner product of x and y in $\mathbb{R}^n$. The uniquely determined measure μ is called the generating measure of Z. We have

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* Corresponding author.
E-mail addresses: lujunguo0301@163.com (L. Guo), lenggangsong@163.com (G. Leng), adu860110@163.com (C. Du).

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$Z = \frac{1}{2}\int_{S^{n-1}}[-y, y]d\mu(y)$, where the integration is done by approximating μ by discrete measures and taking Minkowski sums of segments in $\mathbb{R}^n$. For more details about zonoids we refer to [11,31,36].

The purpose of this article is to discuss the properties of a new operator Z_ϕ which is on Orlicz generalization of the mean zonoid operator of Zhang [37]. Motivated by recent progress in the asymmetric L_p Brunn–Minkowski theory (see e.g. [17–19,21,26,27,33,35]), Lutwak, Yang, and Zhang introduced the Orlicz Brunn–Minkowski theory in two articles [24,25]. Since this seminal work, this new theory has evolved rapidly (see e.g. [5,8,15,16,20,23,34]). We consider even convex functions $\phi : \mathbb{R} \to [0, \infty)$ such that $\phi(0) = 0$. This means that ϕ must be decreasing on $(-\infty, 0]$ and increasing on $[0, \infty)$. We require that ϕ is strictly decreasing on $(-\infty, 0]$ and strictly increasing on $[0, \infty)$. The class of such ϕ will be denoted by $\mathcal{C}$.

Let $K, L \subset \mathbb{R}^n$ be convex bodies (compact convex sets with nonempty interiors) and $\phi \in \mathcal{C}$. We define the Orlicz mean zonoid body $Z_\phi(K, L)$ of K and L as the convex body whose support function at $x \in \mathbb{R}^n$ is given by

$$h_{Z_\phi(K,L)}(x) = \inf\left\{\lambda > 0 : \frac{1}{V(K)V(L)}\int_K\int_L \phi\left(\frac{x \cdot (y - z)}{\lambda}\right)dydz \le 1\right\}, \tag{1.2}$$

where the integration is with respect to Lebesgue measure in $\mathbb{R}^n$.

If $\phi_p(t) = |t|^p$ with $p \ge 1$ and $K = L$ in (1.2), then

$$Z_{\phi_p}(K, K) = Z_pK, \tag{1.3}$$

where Z_pK is the L_p mean zonoid of K, whose support function is given by

$$h^p_{Z_pK}(x) = \frac{1}{V(K)^2}\int_K\int_K \left|x \cdot (y - z)\right|^p dydz. \tag{1.4}$$

For $p = 1$, the body Z_1K is the mean zonoid of K (see, e.g., [37]). We write $\mathcal{K}^n$ for the set of convex bodies in $\mathbb{R}^n$. We write $\mathcal{K}^n_0$ for the set of convex bodies that contain the origin in their interiors. For an arbitrary binary operation $* : (\mathcal{K}^n)^2 \to \mathcal{K}^n$, we say that $*$ is $GL(n)$ covariant, if $A(K * L) = A(K) * A(L)$ for all $A \in GL(n)$, see, e.g., Gardner [7].

Our first main result is as follows.

Theorem 1.1. *If $\phi \in \mathcal{C}$, then the operator Z_ϕ maps $(\mathcal{K}^n)^2$ to $\mathcal{K}^n_0$ and is commutative, homogeneous of degree 1, $GL(n)$ covariant and continuous in each argument with respect to the Hausdorff metric.*

We will also establish the following general affine isoperimetric inequality for convex bodies. Here and in the following, for $K \in \mathcal{K}^n$, we denote by B_K the n-ball with the same volume as K centered at the origin.

Theorem 1.2. *If $K, L \in \mathcal{K}^n_0$ and $\phi \in \mathcal{C}$, then*

$$V\big(Z_\phi(B_K, B_L)\big) \le V\big(Z_\phi(K, L)\big) \tag{1.5}$$

with equality if K and L are dilated ellipsoids having the same midpoints.

While the operator $Z_\phi(K, L)$ is not invariant under independent translations of K and L, the mean zonoid operator and the L_p mean zonoid operator are both translation invariant, i.e., for any $t \in \mathbb{R}^n$, $Z_p(K - t) = Z_pK$. In fact, from (1.4), we have