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Variational principle for topological pressures on subsets

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ABSTRACT

This paper studies the relations between Pesin–Pitskel topological pressure on an arbitrary subset and measure theoretic pressure of Borel probability measures, which extends Feng and Huang's recent result on entropies [13] for pressures. More precisely, this paper defines the measure theoretic pressure $P_\mu(T, f)$ for any Borel probability measure, and shows that $P_B(T, f, K) = \sup\{P_\mu(T, f) : \mu \in \mathcal{M}(X), \mu(K) = 1\}$, where $\mathcal{M}(X)$ is the space of all Borel probability measures, $K \subseteq X$ is a non-empty compact subset and $P_B(T, f, K)$ is the Pesin–Pitskel topological pressure on K. Furthermore, if $Z \subseteq X$ is an analytic subset, then $P_B(T, f, Z) = \sup\{P_B(T, f, K) : K \subseteq Z \text{ is compact}\}$. This paper also shows that Pesin–Pitskel topological pressure can be determined by the measure theoretic pressure.

1. Introduction

Throughout this paper X is a compact metric space with metric d and $T: X \to X$ is a continuous transformation, such a pair (X, T) is a *topological dynamical system* (TDS for short). Let $\mathcal{M}(X)$ be the space of all Borel probability measures on X, and denote by $\mathcal{M}_T$ and $\mathcal{E}_T$ the set of all T-invariant (respectively, ergodic) Borel probability measures on X. For any $\mu \in \mathcal{M}_T$, let $h_\mu(T)$ denote the measure theoretic entropy of μ with respect to T and let $h_{top}(T)$ denote the topological entropy of the system (X, T), see [34] for the precise definitions. It is well-known that entropies constitute essential invariants in the characterization of the complexity of dynamical systems. The classical measure theoretic entropy for an invariant measure and the topological entropy are introduced in [19] and [1] respectively. The basic relation between topological entropy and measure theoretic entropy is the variational principle, e.g., see [34].

Topological pressure is a non-trivial and natural generalization of topological entropy. One of the most fundamental dynamical invariants that associate to a continuous map is the topological pressure with

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a potential function. It roughly measures the orbit complexity of the iterated map on the potential function. Ruelle [29] introduced topological pressure of a continuous function for $\mathbb{Z}^n$-actions on compact spaces and established the variational principle for topological pressure when the action is expansive and satisfies the specification property. Later, Walters [33] generalized the variational principle for a $\mathbb{Z}_+$-action without these assumptions. Misiurewicz [23] gave an elegant proof of the variational principle for $\mathbb{Z}_+^n$-action. See [16,26,25,30–32] for the variational principle for amenable group actions and [11,18] for sofic groups actions. And we would like to mention, Barreira [2–4], Cao, Feng and Huang [8], Mummert [24], Zhao and Cheng [39,40] dealing with variational principle for topological pressure with nonadditive potentials, and Huang and Yi [17] and Zhang [36], where variational principle for the local topological pressure are also considered. This paper conducts research for $\mathbb{Z}$ or $\mathbb{Z}_+$-actions.

From a viewpoint of dimension theory, Pesin and Pitskel' [28] defined the topological pressure on noncompact sets which is a generalization of Bowen's definition of topological entropy on noncompact sets [5], and they proved the variational principle under some supplementary conditions. The notions of the topological pressure, variational principle and equilibrium states play a fundamental role in statistical mechanics, ergodic theory and dynamical systems (see the books [6,34]).

Motivated by Feng and Huang's recent work [13], where the authors studied the variational principle between Bowen topological entropy on an arbitrary subset and measure theoretic entropy for Borel probability measures (not necessarily invariant), recently, Wang and Chen generalized Feng–Huang's result to BS-dimension [35]. As a natural generalization of topological entropy, topological pressure is a quantity which belongs to one of the concepts in the thermodynamic formalism. This study defines measure theoretic pressure for a Borel probability measure and investigates its variational relation with the Pesin–Pitskel topological pressure. Moreover, it is proved that Pesin–Pitskel topological pressure is determined by measure theoretic pressure of Borel probability measures. The outline of the paper is as follows. The main results, as well as those definitions of the measure theoretic pressure and topological pressures, are given in Section 2. The proof of the main results and related propositions are given in Section 3.

2. Definitions and the statement of main results

This section first gives the definition of measure theoretic pressure for any Borel probability measure, and then recalls different kinds of definitions of the topological pressure. The main results of this paper is given in the end of this section.

We first give some necessary notations. For any $n \in \mathbb{N}$ and $\epsilon > 0$, let $d_n(x, y) = \max\{d(T^i(x), T^i(y)) : 0 \le i < n\}$ for any $x, y \in X$ and $B_n(x, \epsilon) = \{y \in X : d_n(x, y) < \epsilon\}$. A set $E \subseteq X$ is said to be an (n, ϵ)-separated subset of X with respect to T if $x, y \in E, x \neq y$, implies $d_n(x, y) > \epsilon$. A set $F \subseteq X$ is said to be an (n, ϵ)-spanning subset of X with respect to T if $\forall x \in X$, $\exists y \in F$ with $d_n(x, y) \le \epsilon$. Let $C(X)$ denote the Banach space of all continuous functions on X equipped with the supremum norm $\|\cdot\|$.

2.1. Measure theoretic pressure

Let $\mu \in \mathcal{M}(X)$ and $f \in C(X)$, the *measure theoretic pressure* of μ for T (w.r.t. f) is defined by

$$P_\mu(T, f) := \int P_\mu(T, f, x)\, \mathrm{d}\mu(x)$$

where $P_\mu(T, f, x) := \lim_{\epsilon\to 0} \liminf_{n\to\infty} (\frac{1}{n} \log[e^{f_n(x)} \cdot \mu(B_n(x, \epsilon))^{-1}])$ and $f_n(x) := \sum_{i=0}^{n-1} f(T^i x)$.

For any $\mu \in \mathcal{M}_T$, using Birkhoff's ergodic theorem (e.g. see [34]) and Brin–Katok's entropy formula [7], for μ-almost every $x \in X$ we have that

$$P_\mu(T, f, x) = h_\mu(T, x) + f^*(x),$$