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A characterization of fast decaying solutions for quasilinear and Wolff type systems with singular coefficients

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ABSTRACT

This paper examines the decay properties of positive solutions for a family of fully nonlinear systems of integral equations containing Wolf potentials and Hardy weights. This class of systems includes examples which are closely related to the Euler–Lagrange equations for the extremal functions of several classical inequalities such as the Hardy–Sobolev and Hardy–Littlewood–Sobolev inequalities. In particular, a complete characterization of the fast decaying ground states in terms of their integrability is provided in that bounded and fast decaying solutions are shown to be equivalent to the integrable solutions. In generating this characterization, additional properties for the integrable solutions, such as their boundedness and optimal integrability, are also established. Furthermore, analogous decay properties for systems of quasilinear equations of the weighted Lane–Emden type are also obtained.

1. Introduction

In this paper, we examine the decay properties of positive solutions at infinity for the following class of integral systems with variable coefficients involving the Wolff potentials and Hardy weights,

$$\begin{cases} u(x) = c_1(x)W_{\beta,\gamma}\big(|y|^{\sigma_1}v^q\big)(x), \\ v(x) = c_2(x)W_{\beta,\gamma}\big(|y|^{\sigma_2}u^p\big)(x). \end{cases} \tag{1.1}$$

Here, the Wolff potential of a function f in $L^1_{loc}(\mathbb{R}^n)$ is defined by

$$W_{\beta,\gamma}(f)(x) = \int_0^\infty \left(\frac{\int_{B_t(x)} f(y)\,dy}{t^{n-\beta\gamma}}\right)^{\frac{1}{\gamma-1}} \frac{dt}{t},$$

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where $n \geq 3$, $\gamma > 1$, $\beta > 0$ with $\beta\gamma < n$, and $B_t(x) \subset \mathbb{R}^n$ denotes the ball of radius t centered at x. Additionally, we take $p, q > 1$, $\sigma_i \leq 0$ and assume the coefficients $c_1(x)$ and $c_2(x)$ are double bounded functions, i.e. there exists a positive constant $C > 0$ such that $C^{-1} \leq c_i(x) \leq C$ for all $x \in \mathbb{R}^n$. The goal of this paper is to determine the necessary and sufficient conditions that completely describe the fast decaying ground states of system (1.1). One motivation for studying the decay properties of solutions for these systems stems from the fact that it is an important ingredient in the classification of solutions and in establishing Liouville type theorems. Another motivation originates from the study of the asymptotic behavior of solutions for elliptic equations. Namely, as we shall discuss below in greater detail, the integral systems we consider are natural generalizations of many elliptic equations, including the weighted equation

$$-\Delta u(x) = |x|^{\sigma} u(x)^p, \quad x \in \mathbb{R}^n,\ \sigma > -2.$$

If $p > \frac{n+\sigma}{n-2}$ so that $n - 2 > \frac{2+\sigma}{p-1}$, the authors in [13,18,21] established that ground states for this equation vanish at infinity with either the slow rate or the fast rate, respectively:

$$u(x) \simeq |x|^{-\frac{2+\sigma}{p-1}} \quad \text{or} \quad u(x) \simeq |x|^{-(n-2)}.$$

Here, the notation $f(x) \simeq |x|^{-\theta}$, where $\theta > 0$, means there exist positive constants c_1 and c_2 such that

$$c_1|x|^{-\theta} \leq f(x) \leq c_2|x|^{-\theta} \quad \text{as } |x| \longrightarrow \infty.$$

Hence, in a sense, our results below extend this example considerably since the Wolff potential has applications to many nonlinear problems and system (1.1) includes several well-known cases. For instance, if $\beta = \alpha/2$ and $\gamma = 2$, the Wolff potential $W_{\beta,\gamma}(\cdot)$ becomes the Riesz potential $I_\alpha(\cdot)$ modulo a constant since

$$\begin{aligned} W_{\frac{\alpha}{2},2}(f)(x) &= \int_0^\infty \frac{\int_{B_t(x)} f(y)\,dy}{t^{n-\alpha}} \frac{dt}{t} = \int_{\mathbb{R}^n} f(y) \left(\int_{|x-y|}^\infty t^{\alpha-n} \frac{dt}{t} \right) dy \\ &= \frac{1}{(n-\alpha)} \int_{\mathbb{R}^n} \frac{f(y)\,dy}{|x-y|^{n-\alpha}} \doteq C(n,\alpha) I_\alpha(f)(x). \end{aligned}$$

Therefore, we can recover from (1.1) a weighted version of the Hardy–Littlewood–Sobolev (HLS) system of integral equations:

$$\begin{cases} u(x) = \displaystyle\int_{\mathbb{R}^n} \frac{|y|^{\sigma_1} v(y)^q}{|x-y|^{n-\alpha}}\,dy, \\ v(x) = \displaystyle\int_{\mathbb{R}^n} \frac{|y|^{\sigma_2} u(y)^p}{|x-y|^{n-\alpha}}\,dy. \end{cases} \tag{1.2}$$

Alternatively, this is sometimes called the Hardy–Sobolev system. If $\sigma_i = 0$ and

$$\frac{1}{1+q} + \frac{1}{1+p} = \frac{n-\alpha}{n},$$

system (1.2) comprises of the Euler–Lagrange equations for a functional associated with the sharp Hardy–Littlewood–Sobolev inequality. In the special case where $p = q = \frac{n+\alpha}{n-\alpha}$, Lieb classified all the maximizers for this functional, thereby obtaining the best constant in the HLS inequality. He then posed the classification of all the critical points of the functional, or the solutions of the integral system, as an open problem [23].