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Uniform and pointwise convergence of Bernstein–Durrmeyer operators with respect to monotone and submodular set functions

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ABSTRACT

We consider the multivariate Bernstein–Durrmeyer operator $M_{n,\mu}$ in terms of the Choquet integral with respect to a monotone and submodular set function μ on the standard d-dimensional simplex. This operator is nonlinear and generalizes the Bernstein–Durrmeyer linear operator with respect to a nonnegative, bounded Borel measure (including the Lebesgue measure). We prove uniform and pointwise convergence of $M_{n,\mu}(f)(x)$ to $f(x)$ as $n \to \infty$, generalizing thus the results obtained in the recent papers [1] and [2].

1. Introduction

Starting from the paper [3], in other three recent papers [1,2,7], uniform, pointwise and L^p convergence (respectively) of $M_{n,\mu}(f)(x)$ to $f(x)$ (as $n \to \infty$) were obtained, where $M_{n,\mu}(f)(x)$ denotes the multivariate Bernstein–Durrmeyer linear operator with respect to a nonnegative, bounded Borel measure μ, defined on the standard simplex

$$S^d = \{(x_1, ..., x_d);\ 0 \le x_1, ..., x_d \le 1,\ 0 \le x_1 + ... + x_d \le 1\},$$

by

$$M_{n,\mu}(f)(x) = \sum_{|\alpha|=n} \frac{\int_{S^d} f(t)B_\alpha(t)d\mu(t)}{\int_{S^d} B_\alpha(t)d\mu(t)} \cdot B_\alpha(x) := \sum_{|\alpha|=n} c(\alpha,\mu) \cdot B_\alpha(x), \quad x \in S^d,\ n \in \mathbb{N}, \tag{1}$$

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where f is supposed to be μ-integrable on S^d. Also, in formula (1), we have denoted $\alpha = (\alpha_0, \alpha_1, ..., \alpha_n)$, with $\alpha_j \geq 0$ for all $j = 0, ..., n$, $|\alpha| = \alpha_0 + \alpha_1 + ... + \alpha_n = n$ and

$$B_\alpha(x) = \frac{n!}{\alpha_0! \cdot \alpha_1! \cdot ... \cdot \alpha_n!}(1 - x_1 - x_2 - ... - x_d)^{\alpha_0} \cdot x_1^{\alpha_1} \cdot ... \cdot x_d^{\alpha_d}$$
$$:= \frac{n!}{\alpha_0! \cdot \alpha_1! \cdot ... \cdot \alpha_n!} \cdot P_\alpha(x).$$

The goal of the present note is to show that the results in [1] and [2] on pointwise and uniform convergence remain valid in the more general setting when μ is a monotone, bounded and submodular set function on S^d and the integrals appearing in the expression of the coefficients $c(\alpha, \mu)$ in formula (1) are Choquet integrals with respect to μ.

2. Preliminaries

In this section we present concepts and results used in the next main section.

Definition 2.1. Let $(\Omega, \mathcal{C})$ be a measurable space, i.e. Ω is a nonempty set and $\mathcal{C}$ is a σ-algebra of subsets in Ω.

(i) (See, e.g., [8, p. 63].) The set function $\mu : \mathcal{C} \to [0, +\infty]$ is called a monotone set function (or capacity) if $\mu(\emptyset) = 0$ and $A, B \in \mathcal{C}$, with $A \subset B$, implies $\mu(A) \leq \mu(B)$. Also, μ is called submodular if

$$\mu(A \cup B) + \mu(A \cap B) \leq \mu(A) + \mu(B), \quad \text{for all } A, B \in \mathcal{C}.$$

If $\mu(\Omega) = 1$, then μ is called normalized.

(ii) (See [4], or [8, p. 233].) Let μ be a normalized, monotone set function defined on $\mathcal{C}$. Recall that $f : \Omega \to \mathbb{R}$ is called $\mathcal{C}$-measurable if for any B, Borel subset in $\mathbb{R}$, we have $f^{-1}(B) \in \mathcal{C}$.

If $f : \Omega \to \mathbb{R}$ is $\mathcal{C}$-measurable, then for any $A \in \mathcal{C}$, the Choquet integral is defined by

$$(C)\int_A f d\mu = \int_0^{+\infty} \mu(F_\beta(f) \cap A) d\beta + \int_{-\infty}^0 [\mu(F_\beta(f) \cap A) - \mu(A)] d\beta,$$

where $F_\beta(f) = \{\omega \in \Omega;\ f(\omega) \geq \beta\}$. If $(C)\int_A f d\mu$ exists in $\mathbb{R}$, then f is called Choquet integrable on A. Note that if $f \geq 0$ on A, then the term integral $\int_{-\infty}^0$ in the above formula becomes equal to zero.

When μ is the Lebesgue measure (i.e. countably additive), then the Choquet integral $(C)\int_A f d\mu$ reduces to the Lebesgue integral.

In what follows, we list some known properties we need for the main section.

Remark 2.2. Let us suppose that $\mu : \mathcal{C} \to [0, +\infty]$ is a monotone set function. Then, the following properties hold:

(i) $(C)\int_A$ is positively homogeneous, i.e. for all $a \geq 0$ we have $(C)\int_A af d\mu = a \cdot (C)\int_A f d\mu$ (for $f \geq 0$ see, e.g., [8, Theorem 11.2 (5), p. 228] and for f of arbitrary sign, see, e.g., [5, p. 64, Proposition 5.1 (ii)]).

(ii) In general, $(C)\int_A (f + g) d\mu \neq (C)\int_A f d\mu + (C)\int_A g d\mu$. However, we have

$$(C)\int_A (f + c) d\mu = (C)\int_A f d\mu + c \cdot \mu(A),$$

for all $c \in \mathbb{R}$ and f of arbitrary sign (see, e.g., [8, pp. 232–233], or [5, p. 65]).