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A multiple iterative splitting method for higher order differential equations

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ABSTRACT

This paper presents an extension of standard iterative splitting schemes to multiple splitting schemes for solving higher order differential equations. The motivation is to solve the systems of equations which occur in the dynamics of electrons in a plasma, using a simplified Boltzmann equation with scattering terms. We reformulate its integro-differential equations into higher order differential equations and then apply iterative splitting methods. Such methods allow decoupling the system into simpler and more quickly solvable first order differential equations: the main idea is to separate the fundamental system of the higher order differential equations into subsystems. This decomposition of the system into first order differential equations allows analyzing such schemes and deriving numerical algorithms. In the numerical part, we discuss the motivation from physical applications in plasma dynamics and present numerical simulations for real-life applications of these integro-differential models.

1. Introduction

The motivation of this paper arose from studying a simulation of active plasma resonance spectroscopy, something which is well established as a plasma diagnostic technique; the first ideas were discussed in [11]. To study such simulation models, we consider an abstract kinetic model based on integro-differential equations, which can be reformulated as higher order differential equations, see the models discussed in [4,18,22]. In this paper, we concentrate on iterative schemes that allow solving such coupled linear and nonlinear systems of equations, see [17]. Historically, splitting schemes were applied to reduce the computational time required, by decoupling the system into simpler and more rapidly solvable equations, see [24,20]. One of the main problems is to reduce the splitting error, see [23], and apply higher order schemes based on a decomposition of products of exponential functions, see [13] and [3], or fixed-point and relaxation schemes to resolve

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kernel functions, see [8]. We consider the second alternative, i.e., fixed-point or relaxation schemes to improve the accuracy of the splitting equations. The main idea in this paper is to decompose a system of integro-differential equations into a system of first order differential equations and apply an iterative splitting scheme in a multiple way as a solver method, see [8]. Such decompositions allow accelerating the solver processes, while the coupling of the separate parts is done with fast iterative splitting methods. Here we discuss a novel iterative method and its theoretical and practical parts.

This paper is organized as follows.

Section 2 presents the model equations and the reduced model for systems of first order differential equations. The splitting schemes are presented and we study their convergence results in Section 3. The results of some numerical benchmark problems and simple real-life experiments are presented in Section 4. Section 5 summarizes the results.

2. Model equations and fundamental systems

Here we specialize to an abstract kinetic model to describe the dynamics of electrons in a plasma which takes into account resonance.

A simplified Boltzmann equation for the electron density is

$$\frac{\partial f(x,v,t)}{\partial t} = -v\cdot\nabla_x f(x,v,t) - \frac{e}{m_e}\nabla_x\phi\cdot\nabla_v f(x,v,t) - \sigma(x,v,t)f(x,v,t) + \int_V \kappa\big(x,v,v'\big)f\big(x,v',t\big)\,dv', \tag{1}$$

$$f(x,v,0) = f_0(x,v), \tag{2}$$

and we apply Dirichlet boundary conditions. The unknown f is the density of the electrons and the initial density f_0 is given. v is the velocity, κ the collision operator, see [22]. Further, e is the charge of the electron and m_e is the mass of the electron. ϕ is the given potential and σ the adsorption or emission parameter of the particles.

We assume that Eq. (1) can be semi-discretized with finite difference or finite volume schemes and by embedding the boundary conditions, we obtain a system of ordinary differential equations, see also [7].

In the following, we will discuss the higher order differential equations in a Banach space $\mathbf{X}_{Real}\subset\mathbb{R}^n$ in the inhomogeneous form (see also [2]):

$$A_0\frac{d^n\mathbf{u}(t)}{d^n t} + A_1\frac{d^{n-1}\mathbf{u}(t)}{dt^{n-1}} + \ldots + A_n = \mathbf{f}(t), \tag{3}$$

$$\frac{d^{i-1}\mathbf{u}(0)}{dt^{i-1}} = \mathbf{u}_{i-1,0}, \quad i=1,\ldots,n, \tag{4}$$

where $A_0,\ldots,A_n : \mathbf{X}_{Real}\to\mathbf{X}_{Real}$ are bounded operators, $\|\cdot\|$ is the corresponding norm in $\mathbf{X}_{Real}$, and $\|\cdot\|_{L(\mathbf{X}_{Real})}$ is the induced operator norm.

Further, the higher order differential equations (3) are given in homogeneous form (see also [2]):

$$A_0\frac{d^n\mathbf{u}(t)}{d^n t} + A_1\frac{d^{n-1}\mathbf{u}(t)}{dt^{n-1}} + \ldots + A_n = 0, \tag{5}$$

$$\frac{d^{i-1}\mathbf{u}(0)}{dt^{i-1}} = \mathbf{u}_{i-1,0}, \quad i=1,\ldots,n. \tag{6}$$

For the transformation, we make the following assumptions: