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Explicit representations of spaces of smooth functions and distributions

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A R T I C L E   I N F O

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A B S T R A C T

We provide an explicit isomorphism between the space of smooth functions $\mathcal{E}(\mathbb{R}^d)$ and its sequence space representation $s \mathbin{\widehat{\otimes}} \mathbb{C}^{\mathbb{N}}$ which isomorphically maps various spaces of smooth functions onto their sequence-space representation, including the space $\mathcal{D}(\mathbb{R}^d)$, of test functions, the space of Schwartz functions $\mathcal{S}(\mathbb{R}^d)$ and the space of "p-integrable smooth functions" $\mathcal{D}_{L^p}(\mathbb{R}^d)$. By restriction and transposition, this isomorphism yields an isomorphism between the space of distributions $\mathcal{D}'(\mathbb{R}^d)$ and its sequence space representation $s' \mathbin{\widehat{\otimes}}_\pi \mathbb{C}^{\mathbb{N}}$ which analogously maps various spaces of distributions isomorphically onto their sequence space representation. We use this isomorphism to construct both a common Schauder basis for these spaces of smooth functions and a common Schauder basis for the corresponding spaces of distributions.

1. Introduction

Starting in 1978, M. Valdivia and D. Vogt discovered sequence space representations for spaces of smooth functions and distributions (see [16–22]). These representations did not only provide isomorphic classifications, but also proved the existence of Schauder bases for many of these spaces.

A main technique in these articles is the Pełczyński decomposition method and hence the representations were not achieved in an explicit manner, i.e., they did not provide explicit formulas for these isomorphisms.

In [9], N. Ortner and P. Wagner presented explicit isomorphisms for spaces of p-integrable smooth functions and distributions, i.e., explicit formulas for isomorphisms $\mathcal{D}_{L^p} \cong s\widehat{\otimes}\ell^p$ and $\mathcal{D}'_{L^p} \cong s'\widehat{\otimes}\ell^p$ for $1 < p < \infty$. In this article they also presented the Valdivia–Vogt structure tables ($1 \le p < q \le \infty$)

$$\begin{array}{ccccccccccccccccccc}
\mathcal{D} & \subset & * & \subset & \mathcal{S} & \subset & \mathcal{D}_{L^1} & \subset & \mathcal{D}_{L^p} & \subset & \mathcal{D}_{L^q} & \subset & \dot{\mathcal{B}} & \subset & \mathcal{D}_{L^\infty} & \subset & \mathcal{O}_C & \subset & \mathcal{O}_M & \subset & \mathcal{E} \\
\wr\Vert & & & & \wr\Vert & & \wr\Vert & & \wr\Vert & & \wr\Vert & & \wr\Vert & & \wr\Vert & & \wr\Vert & & \wr\Vert & & \wr\Vert \\
s \mathbin{\widehat{\otimes}}_\iota \mathbb{C}^{\mathbb{N}} & \subset & s \mathbin{\widehat{\otimes}}_\pi \mathbb{C}^{(\mathbb{N})} & \subset & s \mathbin{\widehat{\otimes}} s & \subset & s \mathbin{\widehat{\otimes}} \ell^1 & \subset & s \mathbin{\widehat{\otimes}} \ell^p & \subset & s \mathbin{\widehat{\otimes}} \ell^q & \subset & s \mathbin{\widehat{\otimes}} c_0 & \subset & s \mathbin{\widehat{\otimes}} \ell^\infty & \subset & s \mathbin{\widehat{\otimes}}_\iota s' & \subset & s \mathbin{\widehat{\otimes}}_\pi s' & \subset & s \mathbin{\widehat{\otimes}} \mathbb{C}^{\mathbb{N}}
\end{array}$$

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(cf. [1, p. 317]) and

$$\begin{array}{ccccccccccccccccccc}
\mathcal{E}' & \subset & \mathcal{O}'_M & \subset & \mathcal{O}'_C & \subset & \mathcal{D}'_{L^1} & \subset & \mathcal{D}'_{L^p} & \subset & \mathcal{D}'_{L^q} & \subset & \dot{\mathcal{B}}' & \subset & \mathcal{D}'_{L^\infty} & \subset & \mathcal{S}' & \subset & \mathcal{D}' \\
\wr\wr & & \wr\wr & & \wr\wr & & \wr\wr & & \wr\wr & & \wr\wr & & & & \wr\wr & & \wr\wr & & \wr\wr \\
s' \mathbin{\widehat{\otimes}} \mathbb{C}^{(\mathbb{N})} & \subset & s' \mathbin{\widehat{\otimes}}_\iota s & \subset & s' \mathbin{\widehat{\otimes}}_\pi s & \subset & s' \mathbin{\widehat{\otimes}} \ell^1 & \subset & s' \mathbin{\widehat{\otimes}} \ell^p & \subset & s' \mathbin{\widehat{\otimes}} \ell^q & \subset & * & \subset & s' \mathbin{\widehat{\otimes}} \ell^\infty & \subset & s' \mathbin{\widehat{\otimes}} s' & \subset & s' \mathbin{\widehat{\otimes}} \mathbb{C}^{\mathbb{N}}
\end{array}.$$

Besides $\mathcal{D}_{L^p}$ and $\mathcal{D}'_{L^p}$ for $1 < p < \infty$, according to N. Ortner and P. Wagner [9], the only spaces in these tables, where an explicit formula for a sequence-space representation was known then, were $\mathcal{S}(\mathbb{R}^d)$ and $\mathcal{S}'(\mathbb{R}^d)$ whose sequence-space representations were found already by L. Schwartz (see [14, p. 262]).

These tables led to the question whether these representations can be achieved in a commutative fashion. In the article [2] the existence of an isomorphism $\Phi\colon \mathcal{E}(\mathbb{R}^d) \to s \mathbin{\widehat{\otimes}} \mathbb{C}^{\mathbb{N}}$ such that the diagram

$$\begin{tikzcd}
\mathcal{D} \arrow[r, hook] \arrow[d, leftrightarrow, "\Phi|_{\mathcal{D}}"] & \mathcal{S} \arrow[r, hook] \arrow[d, leftrightarrow, "\Phi|_{\mathcal{S}}"] & \mathcal{D}_{L^p} \arrow[r, hook] \arrow[d, leftrightarrow, "\Phi|_{\mathcal{D}_{L^p}}"] & \dot{\mathcal{B}} \arrow[r, hook] \arrow[d, leftrightarrow, "\Phi|_{\dot{\mathcal{B}}}"] & \mathcal{D}_{L^\infty} \arrow[r, hook] \arrow[d, leftrightarrow, "\Phi|_{\mathcal{D}_{L^\infty}}"] & \mathcal{O}_C \arrow[r, hook] \arrow[d, leftrightarrow, "\Phi|_{\mathcal{O}_C}"] & \mathcal{O}_M \arrow[r, hook] \arrow[d, leftrightarrow, "\Phi|_{\mathcal{O}_M}"] & \mathcal{E} \arrow[d, leftrightarrow, "\Phi"] \\
s \mathbin{\widehat{\otimes}}_\iota \mathbb{C}^{(\mathbb{N})} \arrow[r, hook] & s \mathbin{\widehat{\otimes}} s \arrow[r, hook] & s \mathbin{\widehat{\otimes}} \ell^p \arrow[r, hook] & s \mathbin{\widehat{\otimes}} c_0 \arrow[r, hook] & s \mathbin{\widehat{\otimes}} \ell^\infty \arrow[r, hook] & s \mathbin{\widehat{\otimes}}_\iota s' \arrow[r, hook] & s \mathbin{\widehat{\otimes}}_\pi s' \arrow[r, hook] & s \mathbin{\widehat{\otimes}} \mathbb{C}^{\mathbb{N}}
\end{tikzcd}$$

is well-defined and hence commutes is shown but also in this case there is no explicit formula given. This commutativity result shows the existence of a common Schauder basis for the separable spaces of smooth functions in the Valdivia–Vogt table of representations of smooth functions, i.e., all of these spaces besides $\mathcal{D}_{L^\infty}$.

Besides the wish for an explicit representation for all these function spaces, the main motivation for the considerations in this article was this common Schauder basis—an object which would be of interest to be known explicitly.

In [3, p. 56] P. Domański states that "even in $\mathcal{D}'(\Omega)$ we have no nice explicit basis known and there is no known explicit isomorphism of $\mathcal{D}'(\Omega)$ and $(s')^{\mathbb{N}}$". At least in the case $\Omega = \mathbb{R}^d$, the explicit version of the isomorphism $\Phi\colon \mathcal{E}(\mathbb{R}^d) \to s \mathbin{\widehat{\otimes}} \mathbb{C}^{\mathbb{N}}$ presented in Section 2 yields by restriction and transposition such an isomorphism and, as the image of the tensor product of the standard bases, a basis for the space $\mathcal{D}'(\mathbb{R}^d)$.

In [2] the existence of the isomorphism Φ is shown by constructing an explicit isomorphism between the space $\mathcal{E}(\mathbb{R}^n)$ of smooth functions and the space $(\mathcal{E}_0)^{\mathbb{Z}^n}$, where $\mathcal{E}_0$ is the space of smooth functions on $[0,1]^n$ which are flat on the set $[0,1]^n \setminus [0,1)^n$. Therefore an explicit isomorphism between $\mathcal{E}_0$ and s provides an explicit isomorphism between $\mathcal{E}(\mathbb{R}^n)$ and $s^{\mathbb{N}}$. In Section 2 we provide such an isomorphism $\Psi\colon \mathcal{E}_0 \cong s$ given by an explicit formula.

In the following we use the Hermite functions H_n as in Example 29.5 in [8, p. 360] and the development of $\mathcal{S}$-functions

$$f = \sum_{n=0}^{\infty} \langle f, H_n \rangle_{L^2} H_n.$$

The Hermite functions are defined as

$$H_n(x) = \frac{(-1)^n}{\sqrt{2^n n! \sqrt{\pi}}} \mathrm{e}^{x^2/2} \left(\frac{\mathrm{d}}{\mathrm{d}x} \right)^n \mathrm{e}^{-x^2}.$$

In this article, we will only use the functions H_{2n}. These functions can be described by the explicit formula

$$H_{2n}(x) = \mathrm{e}^{-\frac{x^2}{2}} \sum_{l=0}^{n} C_{l,n} x^{2l}, \quad C_{l,n} = \frac{(-1)^{n-l}(2n)! 2^{2l-n}}{\sqrt{(2n)! \sqrt{\pi}} (2l)!(n-l)!}. \tag{1}$$