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An algebraic calculation method for the acoustic low frequency expansion ☆

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ABSTRACT

In the present work we propose an analytical method for the algebraic calculation of the low frequency expansion of the solution of a scalar scattering problem upon a smooth starshaped scatterer. The method is based on the far field expansion theorem, introduced by Atkinson and Wilcox in the middle of the last century, applied in the low frequency realm. In this view, the need for solving elliptical boundary value problems to obtain the low frequency approximations is replaced by an analytical procedure, easily encoded in an algorithm including only algebraic operators. As a demonstration example, the method is applied to the acoustic scattering problem with plane wave excitation upon three different non-penetrable spherical scatterers. The first low frequency approximations are deduced, yielding the validity of the proposed method by both recovering already known results and accurately deriving higher orders of approximation.

1. Introduction

The low frequency scattering theory is well known, widely used and taught in most applied mathematics postgraduate study programs. In order to obtain the low frequency expansion of the solution of a particular scattering problem, one needs to solve a sequence of elliptic boundary value problems, since each order of approximation requires the knowledge of all the previous ones [13–15,19].

The analytical treatment of such problems offers accurate results but is restricted to specific geometries and to low orders of approximation. In particular, the scatterer's surface has to be a level surface of a coordinate system amenable to the method of separation of variables, in order to have analytical solutions of the corresponding boundary value problems. Even in this case, the calculation can become inaccessible by analytical means, when high orders of approximation are considered. Therefore, the use of approximate methods becomes inevitable, bringing along numerical errors that propagate within the successive calculations.

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The procedure we propose in the present work is purely analytical and provides accurately all the low frequency approximations of any order. The proposed method intervenes in the low frequency theory at the step of the calculation of the low frequency coefficients of the total field's expansion. It aims to replace the inversion of the differential operator involved in solving the corresponding boundary value problems with simple algebraic multiplication, thus reducing the calculation cost, both in analytical and in numerical treatment, as well as offering a closed form expression for each low frequency coefficient.

The result stems from using the spectral property of the Beltrami differential operator, together with asymptotic analysis in the realm of low frequencies. The basic idea was introduced by Sommerfeld [22] and it has been formulated as an expansion theorem first in a restrictive case by Atkinson [11], extended by Barrar and Kay [12] and completed in its most general form by Wilcox [23].

Their work offered an expansion theorem that provides the far field expansion of a solution of the Helmholtz equation that satisfy the Sommerfeld radiation condition, in terms of inverse powers of the distance. They proved that such expansion converges absolutely and uniformly in the exterior of a sphere circumscribing the scatterer. They also showed that the coefficients of such expansion can all be deduced from the scattering amplitude, or the far field pattern, via a rather simple recurrence relation which includes the Beltrami operator. The resulting expansion will be referred in this work as (AW) expansion, honoring the main contributors for its final form.

As Kleinman has already suggested since 1965 in [17], the (AW) expansion can be used to produce the low frequency scattering coefficients of the corresponding problem. In particular, in [17], as well as in [9], a construction method is developed, by which the solution of a general scattering problem is explicitly expressed in terms of the corresponding potential problem. The (AW) expansion enters the construction method by offering a convenient expression for the scattered field, modified properly to meet the regularity properties needed for the method to apply. This method results to an integro-differential equation, which is solved iteratively for small enough values of the wavenumber. As an illustration, the low frequency expansion of the solution of the Dirichlet and of the Neumann scattering problems for plane wave incidence on a sphere are provided in [17] and in [9], respectively. To our knowledge, no other use of the (AW) expansion is made in obtaining the low frequency scattering coefficients.

Nevertheless, other calculation methods for the full asymptotic expansion of the scattered field in the low frequencies, appear in the literature, as for example in [7] and [6], where variational techniques have been used with respect to the electromagnetic scattering problem and to its scalar quasistatic limit respectively, but the calculations do not avoid the need of the inversion of a differential operator or of the solution of a variational equation. Moreover in [4] it is proved that, the scattering coefficients for the reduced wave equation in the presence of a penetrable scatterer in two dimensions are directly connected to the Fourier coefficients of the problem's scattering amplitude, a feature also dominant in the algebraic method that we propose in this paper.

Rigorous analytic derivation of the full asymptotic expansion of the solution of the Helmholtz equation with respect to a small parameter has been also reported in [2,3]. The derivation is based on boundary layer potential techniques and the resulting series correspond to perturbations due to small inhomogeneities in the medium of propagation. In [2,3,5,8] the importance of having analytical access to high order terms of the asymptotic expansions is pointed out, in order to reconstruct the scatterer's characteristics with a higher resolution, when dealing with the inverse scattering problem.

In [5,8], special interest is paid to the asymptotic expansion of the solution of the full Maxwell's equations in complex domains where small dielectric objects are embedded in the medium of propagation, due to the important practical applications in biomedical imaging and in metal inspection technology. Likewise, it is among our immediate scopes to investigate the applicability of the method we propose in the present work in the electromagnetic scattering problem, based on the corresponding form of the Atkinson–Wilcox expansion [10].