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Note

A note on critical point metrics of the total scalar curvature functional

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ABSTRACT

The aim of this note is to investigate the critical points of the total scalar curvature functional restricted to the space of metrics with constant scalar curvature of unitary volume, for simplicity CPE metrics. In this note, we give a necessary and sufficient condition on the norm of the gradient of the potential function for a CPE metric to be Einstein.

1. Introduction

Let M^n be an n-dimensional compact (without boundary) oriented manifold, and $\mathcal{M}$ be the set of smooth Riemannian structures on M^n of volume 1. Given a metric $g \in \mathcal{M}$ we define the total scalar curvature, or Einstein–Hilbert functional $\mathcal{S} : \mathcal{M} \to \mathbb{R}$ by

$$\mathcal{S}(g) = \int_M R_g dv, \tag{1.1}$$

where R_g and dv stand, respectively, for the scalar curvature of M^n and the volume form determined by the metric and orientation. It is well known that the critical point metrics of the total scalar curvature functional $\mathcal{S}$ restricted to $\mathcal{M}$ are Einstein; for more details see Chapter 4 in [2].

On the other hand, since the solution to the Yamabe problem shows that any compact manifold M^n carries many metrics with constant scalar curvature, we may consider the space of csc-metrics (constant scalar curvature metrics), namely, $\mathcal{C} = \{g \in \mathcal{M} \mid R_g \text{ constant}\}$, on M^n. In fact, a theorem due to Koiso [9] shows that, under generic condition, $\mathcal{C}$ is an infinite dimensional manifold, whose tangent space is $\{h \in C^\infty(S^2M);\ \Delta_g \mathfrak{L}_g h = 0\}$ (cf. Theorem 4.44 in [2, p. 127]). Formally the Euler–Lagrangian equation of Hilbert–Einstein action restricted to $\mathcal{C}$ may be written as the following critical point equation

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$$Ric - \frac{R}{n}g = Hess\, f - \left(Ric - \frac{R}{n-1}g\right)f, \tag{1.2}$$

where Ric, R and $Hess$ stand, respectively, for the Ricci tensor, the scalar curvature and the Hessian form on M^n. For more details see [1] and [2].

It is easy to see that (1.2) yields

$$(1+f)\mathring{Ric} = \nabla^2 f + \frac{Rf}{n(n-1)}g,$$

or, in local coordinates,

$$(1+f)\mathring{R}_{ij} = \nabla_i\nabla_j f + \frac{Rf}{n(n-1)}g_{ij} \tag{1.3}$$

where $\mathring{R}_{ij} = R_{ij} - \frac{R}{n}g_{ij}$ and R_{ij} is the Ricci tensor of metric g_{ij}.

It is worth pointing out that if f is constant in Eq. (1.3), then $f = 0$ and g must be Einstein. We also notice that computing the trace in (1.3) we obtain

$$-\Delta f = \frac{Rf}{(n-1)}. \tag{1.4}$$

In particular, f is an eigenfunction of the Laplacian. Since the Laplacian has non-positive spectrum we may conclude that R must be positive, see [1].

It has been conjectured that the critical points of the total scalar curvature functional $\mathcal{S}$ restricted to $\mathcal{C}$ are Einstein (cf. [2, p. 128]). For this reason, it is of interest to know what are the singular points of the restriction of the Einstein–Hilbert functional to $\mathcal{C}$. Following the notations developed in [1] we consider the following definition.

Definition 1. A CPE metric is a triple (M^n, g, f), where (M^n, g) is a compact oriented Riemannian manifold of dimension $n \geq 3$ with constant scalar curvature and volume 1 while f is a smooth potential satisfying Eq. (1.3).

With this definition the conjecture proposed in [2] may be restated in terms of CPE. The geometric structure of an Einstein solution of (1.3) is known to be simple. Indeed, Obata showed that such a solution is isometric to a standard n-sphere [11]. In the last years, several papers have been published in an attempt to prove the conjecture. For instance, Lafontaine [10] proved that the CPE Conjecture is true under locally conformally flat assumption and

$$Ker\{\nabla_g^2 f - (\Delta_g f)g - fRic_g\} \neq 0 \tag{1.5}$$

and in 2011 Chang, Hwang and Yun avoided the condition (1.5) (cf. [3]). Moreover, Hwang, in [5], considered the conjecture in the 3-dimensional case under the condition that the norm of the gradient of the potential function f was a function of f only; see also [7]. Also, Hwang in [6] proved the CPE Conjecture provided $f \geq -1$. In [4], Yun, Chang and Hwang prove that if the manifold with the critical point metric has parallel Ricci tensor, then it is isometric to a standard sphere. In [3], the conjecture was proved with harmonic curvature presumption. Moreover, in 3-dimensional case the conjecture was proved under the additional assumption (1.5) (Hwang [8]). Qing and Yuan in the article [12] prove the conjecture for 3-dimensional case with the Bach flatness assumption. Recently, Barros and Ribeiro Jr. [1] proved that the CPE Conjecture is also true for 4-dimensional half locally conformally flat. For more details about this problem see [1].

In this paper, we give a necessary and sufficient condition for a CPE metric to be Einstein in any dimension. More precisely, we have the following result.