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# Spreading speed and sharp asymptotic profiles of solutions in free boundary problems for nonlinear advection–diffusion equations

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A B S T R A C T

In this study, we consider free boundary problems for nonlinear advection–diffusion equations of the form $u_t - u_{xx} + \beta u_x = f(u)$ for $t > 0$, $g(t) < x < h(t)$, where $x = g(t)$ and $x = h(t)$ are free boundaries. This problem may be used to describe the spreading of a biological or chemical species where the free boundaries represent the expanding fronts. When $f$ is a logistic nonlinearity, it has been shown that the asymptotic spreading speeds of the two fronts $h(t)$ and $g(t)$ are different due to the advection term. In this study, for monostable, bistable, and combustion types nonlinearities, we give much sharper estimates of the different spreading speeds of the fronts, and we also prove that the solution converges to a semi-wave in $C^2$-norm as $t \to \infty$ when spreading occurs. We develop new approaches and extend a previous result to the problem with the advection term.

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## 1. Introduction and main results

We consider the following free boundary problem for the nonlinear advection–diffusion equation:

$$\begin{cases} u_t - u_{xx} + \beta u_x = f(u), & t > 0,\ g(t) < x < h(t), \\ u(t, g(t)) = u(t, h(t)) = 0, & t > 0, \\ g'(t) = -\mu u_x(t, g(t)), & t > 0, \\ h'(t) = -\mu u_x(t, h(t)), & t > 0, \\ -g(0) = h(0) = h_0, \quad u(0, x) = u_0(x), & -h_0 \le x \le h_0, \end{cases} \tag{1.1}$$

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where $\beta$, $\mu$, and $h_0$ are given positive constants, and $x = h(t)$ and $x = g(t)$ are the moving boundaries that need to be determined together with $u(t, x)$. The initial function $u_0 \in C^2([-h_0, h_0])$ satisfies

$$u_0(-h_0) = u_0(h_0) = 0, \qquad u_0'(-h_0) > 0, \qquad u_0'(h_0) < 0, \qquad u_0(x) > 0 \quad \text{in } (-h_0, h_0).$$

We study the asymptotic behaviors of solutions $(u, g, h)$ to (1.1) with three types of nonlinear terms $f(u)$:

$$(\mathrm{f_M}) \quad \text{monostable case}, \qquad (\mathrm{f_B}) \quad \text{bistable case}, \qquad (\mathrm{f_C}) \quad \text{combustion case}.$$

In the monostable case $(\mathrm{f_M})$, we assume that $f$ is $C^1$ and that it satisfies

$$f(0) = f(1) = 0, \qquad f'(0) > 0, \qquad f'(1) < 0, \qquad (1-u)f(u) > 0 \quad \text{for } u > 0,\ u \neq 1.$$

A typical example of $f$ that satisfies $(\mathrm{f_M})$ is $f(u) = u(1-u)$.

In the bistable case $(\mathrm{f_B})$, we assume that $f$ is $C^1$ and that it satisfies

$$\begin{aligned} &f(0) = f(\theta) = f(1) = 0, \\ &f(u) < 0 \quad \text{in } (0, \theta), \qquad f(u) > 0 \quad \text{in } (\theta, 1), \qquad f(u) < 0 \quad \text{in } (1, \infty) \end{aligned}$$

for some $\theta \in (0, 1)$, $f'(0) < 0$, $f'(1) < 0$ and $\int_0^1 f(s)ds > 0$. The function $f(u) = u(u-\theta)(1-u)$ with $\theta \in \left(0, \frac{1}{2}\right)$ is a typical example of $f$ that satisfies $(\mathrm{f_B})$.

In the combustion case $(\mathrm{f_C})$, we assume that $f$ is $C^1$ and that it satisfies

$$f(u) = 0 \quad \text{in } [0, \theta], \qquad f(u) > 0 \quad \text{in } (\theta, 1), \qquad f'(1) < 0, \qquad f(u) < 0 \quad \text{in } [1, \infty)$$

for some $\theta \in (0, 1)$, and that there exists a small $\delta_0 > 0$ such that

$$f(u) \text{ is nondecreasing in } (\theta, \theta + \delta_0).$$

For these three types of nonlinearities, problem (1.1) may be used to describe the spreading of a biological or chemical species, where the free boundaries $x = g(t)$ and $x = h(t)$ represent the spreading fronts of the species, the density of which is represented by $u(t, x)$. The behaviors of the free boundaries are determined by the Stefan-like conditions which imply that the population pressure at the free boundaries is the driving force of the spreading fronts. We can deduce that this condition is still the same as the problem without the advection term.

Recently, problem (1.1) with $\beta = 0$ was studied in [7]. In particular, for monostable $(\mathrm{f_M})$, bistable $(\mathrm{f_B})$, and combustion type $(\mathrm{f_C})$ nonlinearity, Du and Lou [7] showed that (1.1) has a unique solution, which is defined for all $t > 0$, and as $t \to \infty$, the interval $(g(t), h(t))$ converges to either a finite interval $(g_\infty, h_\infty)$ or $\mathbb{R}$. Moreover, in the former case, $u(t, x) \to 0$ uniformly in $x$ (vanishing), whereas in the latter case, $u(t, x) \to 1$ locally uniformly in $x \in \mathbb{R}$ (spreading), except for a non-generic transition case where $f$ is a bistable $(\mathrm{f_B})$ or combustion type $(\mathrm{f_C})$ nonlinearity. In [15,16], the left boundary was fixed under the Dirichlet boundary condition and the detailed asymptotic behaviors of the solutions were studied.

For problem (1.1) with $\beta = 0$, it was shown by [7] that when spreading occurs, there exists $c^* > 0$ such that

$$\lim_{t\to\infty} \frac{-g(t)}{t} = \lim_{t\to\infty} \frac{h(t)}{t} = c^*. \tag{1.2}$$

Moreover, Du, Matsuzawa, and Zhou [9] obtained a sharper estimate for the spreading speed of the fronts than (1.2) and a convergence result. In particular, [9] showed that when spreading occurs for the solution $(u, g, h)$ of (1.1) with $\beta = 0$, there exist $\hat{H}$, $\hat{G} \in \mathbb{R}$ such that

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