J. Math. Anal. Appl. 428 (2015) 238–257

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Linear independence of compactly supported separable shearlet systems

Jackie Ma [*,1], Philipp Petersen [*,1]

Technische Universität Berlin, Department of Mathematics, Straße des 17. Juni 136, 10623 Berlin, Germany

ARTICLE INFO

Article history:
Received 4 September 2014
Available online 3 March 2015
Submitted by P.G. Lemarie-Rieusset

Keywords:
Frames
Shearlets
Linear independence

ABSTRACT

This paper examines linear independence of shearlet systems. This property has already been studied for wavelets and other systems such as, for instance, for Gabor systems. In fact, for Gabor systems this problem is commonly known as the HRT conjecture. In this paper we present a proof of linear independence of compactly supported separable shearlet systems. For this, we employ a sampling strategy to utilize the structure of an implicitly given underlying oversampled wavelet system as well as the shape of the supports of the shearlet elements.

1. Introduction

Shearlet systems are representation systems that were first introduced by K. Guo, G. Kutyniok, D. Labate, W.-Q Lim and G. Weiss in [14,16,23]. Furthermore, compactly supported separable shearlet systems were introduced in [24,20], where it was also proven that these systems can constitute frames, cf. [5]. In this paper, we study a further structural property of compactly supported separable shearlet systems, namely *linear independence*. The term "linear independence" has to be clarified for this, since in an infinite dimensional space different notions are possible.

Definition 1.1. Let $\{f_i\}_{i\in I}$ be a countable sequence of elements in a Banach space $\mathcal{X}$.

i) If $\sum_{i\in I} c_i f_i = 0$ implies $c_i = 0$ for every $i \in I$, then we call $\{f_i\}_{i\in I}$ *ω-independent*.
ii) If for any finite set $J \subset I$ we have $\sum_{i\in J} c_i f_i = 0$ if and only if $c_i = 0$ for all $i \in J$, then we call $\{f_i\}_{i\in I}$ *linearly independent* (or *finitely linearly independent*).

* Corresponding authors.
E-mail addresses: ma@math.tu-berlin.de (J. Ma), petersen@math.tu-berlin.de (P. Petersen).

[1] DFG Collaborative Research Center TRR 109 "Discretization in Geometry and Dynamics".

http://dx.doi.org/10.1016/j.jmaa.2015.03.001

Note that ω-independence implies linear independence. The question whether certain representation systems are ω-independent or linearly independent, is connected to deep conjectures in harmonic analysis, e.g. the *HRT conjecture* and the *Feichtinger conjecture*. We first explain Feichtinger's conjecture, whereas the HRT conjecture, formulated by C. Heil, J. Ramanathan, and P. Topiwala, will be described in Subsection 1.1.1.

The Feichtinger conjecture, see [4], claims that every bounded frame, i.e. a frame that additionally satisfies $0 < \inf_{i\in I} \|f_i\|_{\mathcal{H}} \leq \sup_{i\in I} \|f_i\|_{\mathcal{H}} < \infty$, can be split into finitely many Riesz sequences, i.e. sequences $(f_i)_{i\in J}, J \subset I$ so that there exist $0 < A_J \leq B_J < \infty$ such that for all $(c_i)_{i\in J} \in \ell^2(J)$ we have

$$A_J \|(c_i)_{i\in J}\|_{\ell^2}^2 \leq \left\| \sum_{i\in J} c_i f_i \right\|_{\mathcal{H}}^2 \leq B_J \|(c_i)_{i\in J}\|_{\ell^2}^2. \tag{1.1}$$

In [4] the Feichtinger conjecture was proven to be equivalent to the *Kadison Singer conjecture*, which in turn has recently been proven by A. Marcus, D.A. Spielman, and N. Srivastava by showing the *paving conjecture*, see [26].

In the next subsection we review some related work in the context of linear independence.

1.1. Related work

One of the first representation systems used in signal and image processing are *Gabor systems*. In this context the question whether the underlying system is linearly independent was posed. To find a general answer to this problem is, however, highly involved and leads to the still open HRT conjecture. The same question was then asked for other representation systems, such as wavelet systems and also localized frames.

1.1.1. Gabor systems

Gabor analysis is build upon time-frequency shifts of a *window function* $g \in L^2(\mathbb{R})$ defined as

$$\pi(x,\omega)g := e^{2\pi i\omega t} g(t-x), \quad (x,\omega) \in \mathbb{R} \times \hat{\mathbb{R}}.$$

For a subset $\Lambda \subset \mathbb{R}^2$ the Gabor system $\mathcal{G}(g,\Lambda)$ is defined as $\mathcal{G}(g,\Lambda) := \{\pi(x,\omega)g, (x,\omega) \in \Lambda\}$, see [11,12]. If Λ is chosen as a countable subset that is "dense enough", the theory of Feichtinger and Gröchenig [9,10] ensures that $\mathcal{G}(g,\Lambda)$ yields a frame for $L^2(\mathbb{R})$.

It has been conjectured in [18], that for every non-zero function $g \in L^2(\mathbb{R})$ and any set of finitely many distinct points $(\alpha_k, \beta_k)_{k=1}^N$ in $\mathbb{R}^2$ the set of functions $\{e^{2\pi\beta_k \cdot} g(\cdot - \alpha_k) : k = 1, \ldots, N\}$ is linearly independent. This conjecture is also called *HRT conjecture*.

While the general claim remains open, there has been a lot of progress in proving the HRT conjecture for a variety of sets Λ and functions g, see also the expository paper [17] and the references therein. For instance, the case where Λ is a *lattice* has been studied. As defined in [18] a lattice in $\mathbb{R}^2$ is any rigid translation of a discrete subgroup of $\mathbb{R}^2$ generated by two linearly independent vectors in $\mathbb{R}^2$. It is a *unit lattice* if every fundamental tile has area 1. A result from [18] states that, if Λ is sampled from a unit lattice, then the Gabor system is linearly independent. For a general lattice the theorem of Linnell [25] guarantees linear independence.

1.1.2. Wavelet systems

A wavelet system is an affine system that is build upon isotropic dilation and translation of generators, so-called *mother wavelets*. We will give a more detailed introduction in Subsection 2.1.

Depending on the generator and the sampling of the parameters, wavelet systems can be constructed such that they constitute a frame, a Riesz basis, or even an orthonormal basis, see [8]. The decomposition of wavelet frames into linearly independent subsystems has been studied, for instance, in [7]. In particular,