J. Math. Anal. Appl. 428 (2015) 258–281

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Asymptotic estimates for the least energy solution of a planar semi-linear Neumann problem

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A R T I C L E I N F O

Article history:
Received 7 October 2014
Available online 17 March 2015
Submitted by M. del Pino

Keywords:
Least energy solution
Semi-linear Neumann boundary condition
Asymptotic estimates
Large exponent

A B S T R A C T

In this work we study the asymptotic behavior of the L^∞ norm of the least energy solution u_p of the following semi-linear Neumman problem

$$\begin{cases} \Delta u = u, \ u > 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = u^p & \text{on } \partial\Omega, \end{cases}$$

where Ω is a smooth bounded domain in $\mathbb{R}^2$. Our main result shows that the L^∞ norm of u_p remains bounded, and bounded away from zero as p goes to infinity, more precisely, we prove that

$$\lim_{p\to\infty} \|u\|_{L^\infty(\partial\Omega)} = \sqrt{e}.$$

1. Introduction

For $\Omega \subset \mathbb{R}^2$ a bounded domain with smooth boundary $\partial\Omega$, we study the least energy solutions to the equation

$$\begin{cases} \Delta u = u, \ u > 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = u^p & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where ν is the outward pointing unit normal vector field on the boundary $\partial\Omega$, and $p > 1$ is a real parameter. We studied this equation in [5], where we showed that for a given integer m, and $p > 1$ large enough, there

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http://dx.doi.org/10.1016/j.jmaa.2015.03.031
0022-247X/

exist at least two solutions U_p to equation

$$\begin{cases} \Delta u = u & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = u^p & \text{on } \partial\Omega, \end{cases} \tag{2}$$

developing m peaks along $\partial\Omega$. More precisely, we prove the existence of m points $\xi_1, \xi_2, \dots, \xi_m \in \partial\Omega$ such that for any $\varepsilon > 0$

$$\|U_p\|_{\Omega\setminus\cup_{i=1}^m B_\varepsilon(\xi_i)} \underset{p\to\infty}{\longrightarrow} 0,$$

and that for each $i = 1, 2, \dots, m$

$$\sup_{\Omega\cap B_\varepsilon(\xi_i)} U_p(x) \underset{p\to\infty}{\longrightarrow} \sqrt{e}.$$

The results in [5, Theorem 1.1] were inspired by the analysis performed in [7], where the authors obtained very similar results for the Dirichlet problem

$$\begin{cases} -\Delta w = w^p & \text{in } \Omega \subset \mathbb{R}^2, \\ \quad w = 0 & \text{on } \partial\Omega. \end{cases} \tag{3}$$

In light of the formal similarity between Eqs. (1) and (3), and the results of Ren and Wei [15,16], and Adimurthi and Grossi [1] about the least energy solutions to Eq. (3) lead us to conjecture in [5] that the least energy solution u_p of Eq. (1) should be bounded, and bounded away from 0, as p tends to infinity, that is, there should exist constants $0 < c_1 \le c_2 < \infty$ such that for all $p > 1$

$$c_1 \le \|u_p\|_{L^\infty(\partial\Omega)} \le c_2, \tag{4}$$

moreover, we conjectured that in fact one should have the following limiting behavior

$$\|u_p\|_{L^\infty(\partial\Omega)} \underset{p\to\infty}{\longrightarrow} \sqrt{e}. \tag{5}$$

Recently, Takahashi [20] has proven (4), in fact he has shown the complete analog of the results of Ren and Wei [15,16] about Eq. (3), in particular, he has shown that u_p looks like a sharp "spike" near a point $x_\infty \in \partial\Omega$, that is [20, Theorem 1]

$$1 \le \liminf_{p\to\infty} \|u_p\|_{L^\infty(\partial\Omega)} \le \limsup_{p\to\infty} \|u_p\|_{L^\infty(\partial\Omega)} \le \sqrt{e}, \tag{6}$$

and [20, Theorem 2]

$$\frac{u_p^p}{\int_{\partial\Omega} u_p^p} \underset{p\to\infty}{\longrightarrow} \delta_{x_\infty} \tag{7}$$

in the sense of measures over $\partial\Omega$. Moreover, the point x_∞ is characterized as a critical point of the Robin function $R(x) = H(x, x)$, where $H(x, y) = G(x, y) + \pi^{-1} \ln |x - y|$ is the regular part of the Green function given by

$$\begin{cases} \Delta_x G(x, y) = G(x, y) & x \in \Omega, \\ \dfrac{\partial G}{\partial \nu_x}(x, y) = \delta_y(x) & x \in \partial\Omega. \end{cases}$$