J. Math. Anal. Appl. 428 (2015) 306–336

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Direct algorithm for multipolar sources reconstruction

Batoul Abdelaziz, Abdellatif El Badia *, Ahmad El Hajj

Sorbonne University, Université de technologie de Compiègne, LMAC Laboratory of Applied Mathematics of Compiègne, Centre de recherche Royallieu - CS 60 319 - 60 203 Compiègne cedex, France

ARTICLE INFO

Article history:
Received 29 April 2014
Available online 14 March 2015
Submitted by X. Zhang

Keywords:
Helmholtz equation
Bioluminescence tomography
EEG/MEG
Multipolar sources
Inverse source problems
Algebraic method

ABSTRACT

This paper proposes an identification algorithm for identifying multipolar sources F in the elliptic equation $\Delta u + \mu u = F$ from boundary measurements. The reconstruction question of this class of sources appears naturally in Helmholtz equation ($\mu > 0$) and in biomedical phenomena particularly in EEG/MEG problems ($\mu = 0$) and bioluminescence tomography (BLT) applications ($\mu < 0$). Previous works have treated the inverse multipolar source problems, only for equations with $\mu = 0$, using algebraic approaches depending on the complex calculation of determinants. Knowing that the novelty in our method concerns several points, the principal one is its simplicity where its proof is not founded on the determinants calculation and its ease in implementation. Moreover, this work involves the general form of equations considering $\mu \in \mathbb{R}$ and at the same time considers a more general type of sources than former related works including sources having small compact support within a finite number of subdomains. Finally, some numerical results are shown to prove the robustness of our identification algorithm.

1. Introduction

In this paper, we consider an inverse source problem whose aim is to reconstruct a source F in the elliptic equation

$$\Delta u + \mu u = F \quad \text{in} \quad \Omega, \tag{1.1}$$

from a single Cauchy data $(f, g) := (u_{|\Gamma}, \frac{\partial u}{\partial \nu}_{|\Gamma})$ prescribed on a sufficiently regular boundary Γ of an open bounded volume $\Omega \subset \mathbb{R}^3$. Here, μ is a fixed real number assumed to be known and ν denotes the outward unit normal to Γ. The main type of sources F considered here, whose motivation is clarified hereinafter, are pointwise multipolar sources defined as

* Corresponding author.
E-mail addresses: batoul.abdelaziz@utc.fr (B. Abdelaziz), abdellatif.elbadia@utc.fr (A. El Badia), ahmad.el-hajj@utc.fr (A. El Hajj).

http://dx.doi.org/10.1016/j.jmaa.2015.03.013
0022-247X/

$$F = \sum_{\ell=1}^{L} \sum_{j=1}^{N^\ell} \sum_{\alpha=0}^{K^\ell} \lambda_{j,\ell}^{\{\alpha_1,\alpha_2,\alpha_3\}} \frac{\partial^\alpha}{\partial_x^{\alpha_1} \partial_y^{\alpha_2} \partial_z^{\alpha_3}} \delta_{\mathbf{S}_j^\ell} \tag{1.2}$$

where $\delta_{\mathbf{S}}$ stands for the Dirac distribution at the point $\mathbf{S}$, the quantities L, N^ℓ, K^ℓ are integers, the coefficients $\lambda_{j,\ell}^{\{\alpha_1,\alpha_2,\alpha_3\}}$ are scalar quantities and $\alpha = \alpha_1 + \alpha_2 + \alpha_3$ with $(\alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}^3$. The points $\mathbf{S}_j^\ell = (x_j^\ell, y_j^\ell, z_j^\ell) \in \Omega$ and the orders of derivation K^ℓ are, respectively, assumed to be mutually distinct. Hence, the inverse problem is brought to the problem of determining the number of sources N^ℓ, their locations $\mathbf{S}_j^\ell$ and the coefficients $\lambda_{j,\ell}^{\{\alpha_1,\alpha_2,\alpha_3\}}$ from a given single Cauchy pair (f, g).

To be more precise, if one defines, for all F of the form (1.2), the following application in $H^{\frac{1}{2}}(\Gamma) \times H^{-\frac{1}{2}}(\Gamma)$

$$\Lambda : F \to (u_{|\Gamma}, \frac{\partial u}{\partial \nu}_{|\Gamma}),$$

then our inverse problem is formulated as follows:

$$\text{Given} \quad (f, g) \in H^{\frac{1}{2}}(\Gamma) \times H^{-\frac{1}{2}}(\Gamma), \quad \text{determine } F \text{ such that} \quad \Lambda(F) = (f, g). \tag{1.3}$$

Physically, a boundary condition in direct problem is imposed and sensors on Γ permit to measure the other quantity related to u so that the Cauchy data $f = u_{|\Gamma}$ and $g = \frac{\partial u}{\partial \nu}_{|\Gamma}$ are obtained.

Using Green's formula with suitably chosen test functions, we can prove, as seen in Subsection 2.2, that the corresponding inverse problem is equivalent to the resolution of the algebraic relationship

$$\alpha_n = \sum_{\ell=1}^{L} \sum_{j=1}^{N^\ell} \sum_{\beta=0}^{K^\ell} \nu_{j,\ell}^{\beta} \tbinom{n}{\beta} (P_j^\ell)^{n-\beta} \tag{1.4}$$

where P_j^ℓ are 2D projections of the locations $\mathbf{S}_j^\ell$ on a certain complex plane, $\tbinom{n}{\beta}$ are the binomial coefficients and $\nu_{j,\ell}$ are coefficients dependant on μ and $\lambda_{j,\ell}$.

In addition to the case $\mu > 0$ that corresponds to the classical Helmholtz equation, this inverse source problem in the mentioned particular framework has several practical motivations especially in certain non-invasive biomedical imaging techniques. More precisely, for $\mu = 0$, one of the important applications is the inverse electroencephalography/magnetoencephalography (EEG/MEG) problem [14,17]. The aim of this problem, used in the epilepsy disease treatment, is to obtain a fairly accurate localization of the epileptogenic sources using electrical and magnetic measures on the scalp. On the other hand, in the case where $\mu < 0$, one of the recent related developing problems is the inverse source problem of the bioluminescence tomography (BLT). In fact, BLT [23] consists in determining an internal bioluminescent source distribution generated by luciferase inducted by reporter genes from external optical measurements. It is an increasingly important tool for biomedical researchers that can help diagnose diseases and evaluate and monitor therapies by allowing real time tomographic localization of the disease's foci. In both applications, these foci and their distribution are described mathematically as sources having small compact support within a finite number of subdomains, namely

$$F = \sum_{j=1}^{N} h_j \chi_{D_j} \quad \text{with} \quad D_j = \mathbf{S}_j + \varepsilon B_j \tag{1.5}$$

where D_j, see Fig. 1, represent the desired foci with $B_j \subset \mathbb{R}^3$ being bounded domains containing the origin, the points $\mathbf{S}_j = (x_j, y_j, z_j) \in \Omega$ and ε is a positive real number less than or equal to 1. Their densities h_j are functions belonging to the space $L^1(\Omega)$.