J. Math. Anal. Appl. 428 (2015) 471–489

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

More elementary operators that are spectrally bounded

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ARTICLE INFO

Article history:
Received 22 December 2014
Available online 17 March 2015
Submitted by R. Timoney

Dedicated to the memory of T.T. West

Keywords:
Elementary operator
Quasi-nilpotent
Spectrally bounded

ABSTRACT

We discuss some necessary and some sufficient conditions for an elementary operator $x \mapsto \sum_{i=1}^{n} a_i x b_i$ on a Banach algebra A to be spectrally bounded. In the case of length three, we obtain a complete characterisation when A acts irreducibly on a Banach space of dimension greater than three.

1. Introduction

Let A and B be unital Banach algebras over the complex field $\mathbb{C}$. Let $r(x)$ denote the spectral radius of an element x in A or B. We say a linear mapping $T: A \to B$ is *spectrally bounded* if, for some constant $M \geq 0$ and all $a \in A$, the estimate $r(Ta) \leq M\, r(a)$ holds. This concept, together with its relatives *spectrally isometric* (i.e., $r(Ta) = r(a)$ for all $a \in A$) and *spectrally infinitesimal* (i.e., $r(Ta) = 0$ for all $a \in A$), was introduced in [17] in order to initiate a systematic investigation of mappings that had, on and off, been discussed in the literature; see, e.g., [2] or [25]. A number of fundamental properties of spectrally bounded operators can be found in [19] and [21] while [20] contains a structure theorem for such operators defined on properly infinite von Neumann algebras. Spectrally bounded operators also appear in connection with the noncommutative Singer–Wermer conjecture and with Kaplansky's problem on invertibility-preserving operators; for details see [18].

An *elementary operator* on A is a bounded linear operator $S: A \to A$ that can be written in the form $Sx = \sum_{i=1}^{n} a_i x b_i$, $x \in A$ for some $a_1, \ldots, a_n$, $b_1, \ldots, b_n \in A$. These operators appear quite naturally in

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http://dx.doi.org/10.1016/j.jmaa.2015.03.012

many contexts; for instance, if A is finite dimensional and semisimple, every linear mapping is of this form. In general, additional assumptions on the algebra and on an operator S may "force" the operator to be elementary: a typical example is the innerness of a derivation $d: A \to A$, that is, $dx = ax - xa$ for some $a \in A$. Properties of elementary operators have been studied under a vast variety of aspects; we refer the reader to [16] and [12] for an overview.

Despite the rich literature on spectrally bounded operators in general, and spectral isometries in particular, see, e.g., [3,6,9,10,18,22,26] and the references contained therein, the supply of examples is still somewhat limited. It is thus close at hand to ask which elementary operators are spectrally bounded, as these operators are given in a more concrete form. Continuing our work started in [4], we aim to provide further answers to this question in the present paper. We shall discuss a number of necessary conditions which, in the case of length three, turn out to be sufficient too. Our new approach exploits the relation with locally quasi-nilpotent elementary operators which, in the algebraic setting, were studied in [5]; in fact, that paper should be read in conjunction with the present one.

In order to illustrate the ideas, let us assume that the elementary operator $S: A \to A$ is spectrally infinitesimal. Let ϱ be an irreducible representation of A on a Banach space E. Since S induces an elementary operator $S_\varrho: \varrho(A) \to \varrho(A)$ via $S_\varrho \circ \varrho = \varrho \circ S$, S_ϱ is spectrally infinitesimal too. By Jacobson's density theorem [2, Theorem 4.2.5], $\varrho(A)$ is a dense (i.e., n-transitive for all n) algebra on E, and we can apply the setting of [5]. Suppose $\zeta \in E$ and $x \in A$ are such that $\varrho(x)\varrho(b_i a_j)\zeta \subseteq \mathbb{C}\zeta$ for all i, j. It is easy to see that this implies that

$$S_\varrho \varrho(x)\bigl(\operatorname{span}\{\varrho(a_1)\zeta, \dots, \varrho(a_n)\zeta\}\bigr) \subseteq \operatorname{span}\{\varrho(a_1)\zeta, \dots, \varrho(a_n)\zeta\};$$

consequently, the restriction of $S_\varrho \varrho(x)$ to the finite-dimensional invariant subspace $\operatorname{span}\{\varrho(a_1)\zeta, \dots, \varrho(a_n)\zeta\}$ has to be nilpotent which then allows us to apply the theory developed in [5]. A first application of this method is presented in Proposition 3.1 and elaborations on this idea provide the main techniques for Section 3; see, in particular, Lemma 3.3.

The general questions that we pursue in this context are as follows. Let the elementary operator $Sx = \sum_{i=1}^n a_i x b_i$, $x \in A$ on A be given.

(a) Suppose S is spectrally bounded.
  (i) What properties of the coefficients a_i, b_i can we derive?
  (ii) Can we find an "improved" representation of S in the sense that the new coefficients have better properties?
(b) Which conditions on the coefficients a_i, b_i ensure that S is spectrally bounded?

After collecting a number of basic properties and tools in Section 2, we give several answers to question (a) above in Section 3 culminating in Theorem 3.6 which describes the size of various spaces associated to the coefficients of the induced elementary operator in an irreducible representation of A in terms of the local dimension. Specialising to the case of length two elementary operators we derive further properties at the end of this section and also correct a small oversight in [4, Theorem 3.5] concerning an exceptional case that can appear in dimension two.

A full answer to both questions (a) and (b) is, at present, only available for elementary operators of short length. It was given in [4] for length two and is provided for length three under the assumption that A acts irreducibly on a Banach space of dimension greater than three in Section 4 below (spaces with smaller dimension need to be treated separately). The formulation seems too technical to allow an extension to the general case so far.