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Criteria for exponential dichotomy for triangular systems

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ABSTRACT

We study the exponential dichotomy properties of nonautonomous systems of linear differential equations. Any such system is kinematically similar to a triangular system. Since it is easy to determine whether or not a diagonal system has an exponential dichotomy, it is important to study the relation between the exponential dichotomy properties of the triangular system and its diagonal part. Without loss of generality, we consider block upper triangular systems and study the relation with their diagonal parts. Our study addresses both bounded and unbounded systems, on the whole line and a half line. We conclude with a study of the spectral properties of bounded systems, focusing on the relation between the whole line spectrum and the half line spectra.

1. Introduction

In this article we study the exponential dichotomy properties of nonautonomous systems of linear differential equations. It is well known that any such system can be transformed by a kinematic similarity into an upper triangular system. Since exponential dichotomy is preserved by kinematic similarity, this means that there is no loss of generality in studying this notion purely in the context of upper triangular systems. Indeed Dieci and Van Vleck [3] study Sacker–Sell spectra in this context. It turns out that the diagonal entries play a key role. So it is important to study the relation between the properties of a triangular system and properties of its diagonal part.

Exponential dichotomy is essentially the same as hyperbolicity. The main difference is that, at least originally, exponential dichotomy pertained to single differential or difference equations whereas hyperbolicity pertained to invariant sets of diffeomorphisms. Now they are essentially different names for the same idea.

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Definition. We say the linear system $\dot{u} = D(t)u$, where $D(t)$ is piecewise continuous, has an *exponential dichotomy* on an interval J (usually $\mathbb{R}$, $\mathbb{R}_+$ or $\mathbb{R}_-$) with projections $P(t)$, constant $k \geq 1$ and exponents $\alpha, \beta > 0$ if the transition matrix $U(t,s)$ satisfies the invariance property

$$P(t)U(t,s) = U(t,s)P(s) \quad \text{if } s,t \in J$$

and the inequalities

$$\begin{aligned} \|U(t,s)P(s)\| &\leq ke^{-\alpha(t-s)} && \text{if } s \leq t \in J \\ \|U(t,s)[\mathbf{I} - P(s)]\| &\leq ke^{\beta(t-s)} && \text{if } t \leq s \in J. \end{aligned}$$

Note that it follows from the invariance that $P(t) = U(t,0)P(0)U(0,t)$. Sometimes we say that the equation has a dichotomy with projection P where $P = P(0)$. If $P = \mathbf{I}$ we only have the first inequality and when $P = 0$ only the second inequality.

Exponential dichotomy generalizes the idea of uniform asymptotic stability to the conditionally stable case. It is an idea going back to Perron [7]. An autonomous system $\dot{x} = Dx$ has an exponential dichotomy if and only if all the eigenvalues of D have nonzero real parts; a periodic system $\dot{x} = D(t)x$ has an exponential dichotomy if and only if all the Floquet exponents have nonzero real parts. A scalar equation $\dot{x} = a(t)x$ has an exponential dichotomy if and only if

$$\limsup_{t-s\to\infty} \frac{1}{t-s}\int_s^t a(u)du < 0 \quad \text{or} \quad \liminf_{t-s\to\infty} \frac{1}{t-s}\int_s^t a(u)du > 0.$$

A diagonal system has an exponential dichotomy if and only if each component scalar equation has. For a general time-dependent system, the eigenvalues of $D(t)$ tell us nothing about the system's dichotomy properties. However if the absolute values of the real parts of the eigenvalues are bounded below by a positive number and $D(t)$ is slowly varying, then the system does have an exponential dichotomy (Coppel [2]). Another criterion is that of diagonal dominance (Lazer [4], Coppel [2]): if $D(t)$ is diagonally dominant in a uniform way, then $\dot{x} = D(t)x$ has an exponential dichotomy. There are also other criteria for exponential dichotomy in terms of the existence of solutions of inhomogeneous equations in certain spaces (Massera and Schäffer [5], Coppel [2]) or in terms of the existence of a Lyapunov function (Coppel [2]) but these are not practical criteria to determine whether or not a given system has a dichotomy. If $\dot{x} = D(t)x$ has an exponential dichotomy, then a small perturbation $\dot{x} = [D(t) + E(t)]x$ has an exponential dichotomy (Coppel [2]). It should be mentioned here that some of these results have been extended to not necessarily invertible difference equations and to equations in Banach spaces (Pötzsche [8]); also the definition has been weakened in certain ways such as nonuniform dichotomies (Barreira and Valls [1]). Our focus here is on exponential dichotomies for differential equations in finite-dimensional spaces; however let us just mention that it seems that our results could be extended quite easily to invertible difference equations in finite-dimensional spaces. (Note that the references given here represent only a small sample of the significant contributions to this field.)

So we study nonautonomous systems of linear differential equations which are in block triangular form (which we may assume without loss of generality are upper triangular). More precisely, we study the relation between the dichotomy and spectral properties of the system and its block diagonal part, both on half lines and on the whole line. What is known already is that a bounded block triangular system has an exponential dichotomy on a half line if and only if its diagonal part has. For the whole line $\mathbb{R}$, when the off diagonal part is bounded, it is known that if the block diagonal part of the system has an exponential dichotomy on $\mathbb{R}$, then the block triangular system has an exponential dichotomy on $\mathbb{R}$ (Palmer [6]).