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Monotonicity criterion for the quotient of power series with applications

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A B S T R A C T

In this paper, we present the necessary and sufficient condition for the monotonicity of the quotient of power series. As applications, some gaps and misquotations in certain published articles are pointed out and corrected, and some known results involving the Landen inequalities for zero-balanced hypergeometric functions are improved.

1. Introduction

In 1955, Biernacki and J. Krzyż [12] (see also [16, Lemma 2.1], [15]) found an important criterion for the monotonicity of the quotient of power series as follows.

Theorem 1.1. *(See [12].) Let $A(t) = \sum_{k=0}^{\infty} a_k t^k$ and $B(t) = \sum_{k=0}^{\infty} b_k t^k$ be two real power series converging on $(-r, r)$ $(r > 0)$ with $b_k > 0$ for all k. If the non-constant sequence $\{a_k/b_k\}$ is increasing (resp. decreasing) for all k, then the function $t \mapsto A(t)/B(t)$ is strictly increasing (resp. decreasing) on $(0, r)$.*

Theorem 1.1 has been widely used to find analytic inequalities [2,5,7–9,22,24,27,28].

The polynomial version of Theorem 1.1 can be found in the literature [15].

Theorem 1.2. *(See [15, Theorem 4.4].) Let $A_n(t) = \sum_{k=0}^{n} a_k t^k$ and $B_n(t) = \sum_{k=0}^{n} b_k t^k$ be two real polynomials with $b_k > 0$ for all k. If the sequence $\{a_k/b_k\}$ is increasing (decreasing), then so is the function $t \mapsto A_n(t)/B_n(t)$ for all $t > 0$.*

More results involving the quotient of power series can be found in [6] and the references therein.

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Next, we deal with the case of the sequence $\{a_k/b_k\}$ being piecewise monotone, that is, for certain $m \in \mathbb{N}$, $\{a_k/b_k\}$ is increasing (resp. decreasing) for $0 \le k \le m$ and decreasing (resp. increasing) for $k \ge m$. In this case, how to determine the monotonicity of the function $t \mapsto A(t)/B(t)$? In 2007, Belzunce, Ortega and Ruiz [11] offered a criterion when $A(t)$ and $B(t)$ have the radius of convergence $R = \infty$, but without giving the details of the proof.

Proposition 1.3. *(See [11, Lemma 6.4].) Suppose that* $\{a_k/b_k, k = 0, 1, \ldots\}$ *is a real non-constant sequence with the property that*

$$\frac{a_k}{b_k}\begin{cases} \text{is increasing in } k, & \text{for } k \le k_0, \\ \text{is decreasing in } k, & \text{for } k \ge k_0 \end{cases}$$

for some positive integer k_0*, also assume that the power series defined by* $A(t) = \sum_{k=0}^{\infty} a_k t^k$ *and* $B(t) = \sum_{k=0}^{\infty} b_k t^k$, $t \ge 0$*, converge absolutely for all* $t \ge 0$*. Then there exists a real number* $t_0 > 0$ *such that*

$$\frac{A(t)}{B(t)}\begin{cases} \text{is strictly increasing in } t, & \text{for } \; t < t_0, \\ \text{is strictly decreasing in } t, & \text{for } \; t \ge t_0. \end{cases}$$

In [10], Proposition 1.3 was rewritten by Baricz, Vesti and Vuorinen in different form as follows.

Proposition 1.4. *(See [10, Theorem 4.6].) Suppose that the power series* $f(x) = \sum_{n\ge 0} a_n x^n$ *and* $g(x) = \sum_{n\ge 0} b_n x^n$ *have the radius of convergence* $r > 0$*, and* $b_n > 0$ *for all* $n \in \{0, 1, 2, \ldots\}$*. If the sequence* $\{a_n/b_n\}$ *satisfies* $a_0/b_0 \le a_1/b_1 \le \cdots \le a_{n_0}/b_{n_0}$ *and* $a_{n_0}/b_{n_0} \ge a_{n_0+1}/b_{n_0+1} \ge \cdots \ge a_n/b_n \ge \cdots$ *for some* $n_0 \ge 1$*, then there exists* $x_0 \in (0, r)$ *such that the function* $x \mapsto f(x)/g(x)$ *is increasing on* $(0, x_0)$ *and decreasing on* (x_0, r)*.*

Proposition 1.3 was also rewritten by Simić and Vuorinen [19] in another different form.

Proposition 1.5. *(See [19, Lemma 1.1(3)].) Suppose that the power series* $f(x) = \sum_{n\ge 0} a_n x^n$ *and* $g(x) = \sum_{n\ge 0} b_n x^n$ *have the radius of convergence* $r > 0$ *and* $b_n > 0$ *for all* $n \in \{0, 1, 2, \ldots\}$*. If the sequence* $\{a_n/b_n\}$ *is monotone increasing (resp. decreasing) for* $0 < n \le n_0$ *and monotone decreasing (resp. increasing) for* $n > n_0$*, then there exists* $x_0 \in (0, r)$ *such that* $f(x)/g(x)$ *is increasing (resp. decreasing) on* $(0, x_0)$ *and decreasing (resp. increasing) on* (x_0, r)*.*

Recently, Proposition 1.5 has been cited and used in the literature [26, Lemma 2.1], [25, Lemma 1], [17, Lemma 6] and [20, Lemma 2.2].

Since without giving the details in the proof of Proposition 1.3, its correctness is questionable. Moreover, Propositions 1.4 and 1.5 may be not valid even if Proposition 1.3 is true, since Proposition 1.3 only contains the case of the radius of convergence $r = \infty$, but both Proposition 1.4 and Proposition 1.5 also contain the case of the radius of convergence $r \in (0, \infty)$. In fact, we easily find the following counter-examples to refute Propositions 1.4 and 1.5.

Example 1.6. Let

$$f(x) = 3 + \sum_{k=1}^{\infty} \frac{2^{2k} - 2}{(2k)!} |B_{2k}| x^k \quad \text{and} \quad g(x) = 1 + \sum_{k=1}^{\infty} \frac{2^{2k}}{(2k)!} |B_{2k}| x^k,$$

where B_k is the Bernoulli number. Then we clearly see that $f(x)$ and $g(x)$ have the radius of convergence $R = \pi^2$. Making use of the well-known formulas [13, pp. 227–229]