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Note

A note on measure-expansive diffeomorphisms

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A B S T R A C T

In this note we prove that a homeomorphism is countably-expansive if and only if it is measure-expansive. This result is applied for showing that the C^1-interior of the sets of expansive, measure-expansive and continuum-wise expansive C^1-diffeomorphisms coincide.

1. Introduction

The phenomenon of expansiveness occurs when the trajectories of nearby points are separated by the dynamical system. The first research that considered expansivity in dynamical systems was by Utz [25]. There, he defined the notion of unstable homeomorphism. An extensive literature related to properties of expansiveness can be found in [1–3,5–10,12,13,15,17–20,23,24,26,27].

If $f: M \to M$ is a homeomorphism of a compact metric space (M, dist) and if $\delta > 0$ we define

$$\Gamma_\delta(x) = \{y \in M : \mathrm{dist}(f^n(x), f^n(y)) \le \delta \text{ for all } n \in \mathbb{Z}\}.$$

Let us recall some definitions that can be found for example in [16]. We say that f is *expansive* if there is $\delta > 0$ such that $\Gamma_\delta(x) = \{x\}$ for all $x \in M$. Given a Borel probability measure μ on M we say that f is *μ-expansive* if there is $\delta > 0$ such that for all $x \in M$ it holds that $\mu(\Gamma_\delta(x)) = 0$. In this case we also say that μ is an *expansive measure* for f. We say that f is *measure-expansive* if it is μ-expansive for every non-atomic Borel probability measure μ. Recall that μ is non-atomic if $\mu(\{x\}) = 0$ for all $x \in M$. The corresponding

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concepts for flows have been considered in [4]. Moreover, we say that f is *countably-expansive* if there is $\delta > 0$ such that for all $x \in M$ the set $\Gamma_\delta(x)$ is countable.

In [16] it is proved that the following statements are equivalent:

1. f is countably-expansive,
2. every non-atomic Borel probability measure of M is expansive with a common expansive constant.

Moreover, they put the following question: are there measure-expansive homeomorphisms of compact metric space which are not countably-expansive? We give a negative answer in Theorem 2.1.

Next we study robust expansiveness of C^1-diffeomorphisms of a smooth manifold. For a fixed manifold M, we denote by $\mathcal{E}$ the set of all expansive diffeomorphisms of M. In order to state our next result let us recall more definitions. We say that $C \subset M$ is a *continuum* if it is compact and connected. A *trivial continuum* (or *singleton*) is a continuum with only one point. Recall from [11,12] that f is *continuum-wise expansive* (or *cw-expansive*) if there is $\delta > 0$ such that if $C \subset M$ is a non-trivial continuum then there is $n \in \mathbb{Z}$ such that $\operatorname{diam}(f^n(C)) > \delta$. Denote by $\mathcal{CE}$ the set of all cw-expansive diffeomorphisms and by $\mathcal{PE}$ the set of all measure-expansive diffeomorphisms of M. We denote by $\operatorname{int} A$ the C^1-interior of a set A of C^1-diffeomorphisms of M. In [14] R. Mañé proved that the C^1-interior of the set of expansive diffeomorphisms coincides with the set of quasi-Anosov diffeomorphisms. See [14] for the definitions and the proof. This result was later extended for cw-expansive homeomorphisms in [21] proving that $\operatorname{int}\mathcal{E} = \operatorname{int}\mathcal{CE}$. Recently, it was proved in [22] that $\operatorname{int}\mathcal{E} = \operatorname{int}\mathcal{PE}$. In Theorem 2.4 we give a new proof of the cited result from [22] based on Theorem 2.1 and [21].

2. Proofs of the results

Our first result holds for a homeomorphism $f\colon M \to M$ of a compact metric space (M, dist).

Theorem 2.1. *The following statements are equivalent:*

1. *f is countably-expansive,*
2. *f is measure-expansive.*

Proof. *Direct.* Let $\delta > 0$ be such that for all $x \in M$ it holds that $\Gamma_\delta(x)$ is countable. Let μ be a non-atomic Borel probability measure. Since μ is non-atomic, by σ-additivity we have that $\mu(\Gamma_\delta(x)) = 0$. Therefore, f is measure-expansive.

Converse. Arguing by contradiction, we assume that f is measure-expansive but there are sequences $\delta_n \to 0$ and $x_n \in M$ such that $\Gamma_{\delta_n}(x_n)$ is uncountable for each $n \geq 1$. As in [16], for each $n \geq 1$ consider a non-atomic Borel probability measure μ_n such that $\mu_n(\Gamma_{\delta_n}(x_n)) = 1$. Consider the Borel probability measure μ defined for a Borel set $A \subset M$ as

$$\mu(A) = \sum_{n=1}^{\infty} \frac{\mu_n(A)}{2^n}.$$

Since every μ_n is non-atomic, we have that μ is non-atomic too. Thus, since f is measure-expansive, there is $\delta > 0$ such that $\mu(\Gamma_\delta(x)) = 0$ for all $x \in M$. Since $\delta_n \to 0$ we can take $\delta_n < \delta$. Then

$$\mu(\Gamma_\delta(x_n)) \geq \mu(\Gamma_{\delta_n}(x_n)) \geq \frac{\mu_n(\Gamma_{\delta_n}(x_n))}{2^n} > 0.$$

This contradiction proves the theorem. □