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The relative Morse index theory for infinite dimensional Hamiltonian systems with applications

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ABSTRACT

We established an index theory for a kind of infinite dimensional linear Hamiltonian system. As applications, we considered the existence and multiplicity of periodic solution for some nonlinear infinite dimensional Hamiltonian systems.

1. Introduction and main results

In this paper, we will consider the following infinite dimensional Hamiltonian system

$$\begin{cases} \partial_t u - \Delta_x u = H_v(t,x,u,v), \\ -\partial_t v - \Delta_x v = H_u(t,x,u,v), \end{cases} \quad \forall (t,x) \in \mathbb{R}\times\Omega, \tag{HS}$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 1$ is a bounded domain with smooth boundary $\partial\Omega$ and $H : \mathbb{R}\times\overline{\Omega}\times\mathbb{R}^{2m} \to \mathbb{R}$ is a C^1 function, $\partial_t := \frac{\partial}{\partial t}$, $\Delta_x := \sum_{i=1}^N \frac{\partial^2}{\partial x_i^2}$, $H_u := \frac{\partial H}{\partial u}$ and $H_v := \frac{\partial H}{\partial v}$. System like (HS) is called unbounded Hamiltonian system, cf. Barbu [3], or infinite dimensional Hamiltonian system, cf. [4,14,15]. This systems arises in optimal control of systems governed by partial differential equations. See, e.g., Lions [20], where the combination of the model $\partial_t - \Delta_x$ and its adjoint $-\partial_t - \Delta_x$ acts as a system for studying the control. Brézis and Nirenberg [6] considered a special case of the system (HS):

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$$\begin{cases} \partial_t u - \Delta_x u = -v^5 + f, \\ -\partial_t v - \Delta_x v = u^3 + g, \end{cases} \tag{1.1}$$

where $f, g \in L^\infty(\Omega)$, subject to the boundary condition $z(t, \cdot)|_{\partial\Omega} = 0$ on variable x and the periodicity condition $z(0, \cdot) = z(T, \cdot) = 0$ on variable t for a given $T > 0$, where $z = (u, v)$. They obtained a solution z with $u \in L^4$ and $v \in L^6$ by using Schauder's fixed point theorem. Clément, Felemer and Mitidieri considered in [14] and [15] the following system which is also a special case of (HS):

$$\begin{cases} \partial_t u - \Delta_x u = |v|^{q-2}v, \\ -\partial_t v - \Delta_x v = |u|^{p-2}u, \end{cases} \tag{1.2}$$

with $\frac{N}{N+2} < \frac{1}{p} + \frac{1}{q} < 1$. Using their variational setting of Mountain Pass type, they proved that there is a $T_0 > 0$ such that, for each $T > T_0$, (1.2) has at least one positive solution $z_T = (u_T, v_T)$ satisfying the boundary condition $z_T(t, \cdot)|_{\partial\Omega} = 0$ for all $t \in (-T, T)$ and the periodicity condition $z_T(T, \cdot) = z_T(-T, \cdot)$ for all $x \in \overline{\Omega}$. Moreover, by passing to limit as $T \to \infty$ they obtained a positive homoclinic solution of (1.2). If the Hamiltonian function H in (HS) can be displayed in the following form

$$H(t, x, u, v) = F(t, x, u, v) - V(x)uv, \tag{1.3}$$

where $V \in C(\Omega, \mathbb{R})$, $H \in C^1(\mathbb{R} \times \overline{\Omega} \times \mathbb{R}^{2m}, \mathbb{R})$, the system (HS) will be rewritten as

$$\begin{cases} \partial_t u + (-\Delta_x + V(x))u = F_v(t, x, u, v), \\ -\partial_t v + (-\Delta_x + V(x))v = F_u(t, x, u, v). \end{cases} \tag{$HS.1$}$$

Bartsch and Ding [4] dealt with the system $(HS.1)$. They established existence and multiplicity of homoclinic solutions of the type $z(t, x) \to 0$ as $|t| + |x| \to \infty$ if $\Omega = \mathbb{R}^N$ with the functions V and F being periodic in variables (t, x) and F being super-quadratic in variable (u, v) and the type of $z(t, x) \to 0$ as $|t| \to \infty$ and $z(t, \cdot)|_{\partial\Omega} = 0$ if Ω is bounded. Recently, there were several results on system (HS) and $(HS.1)$, cf. [16,26,29–31].

In this paper we are mainly interested in the asymptotically linear case of system (HS) with the Hamiltonian function $H(t, x, u, v)$ being T-periodic in the first variable and asymptotically linear in variable (u, v) as $|z| \to \infty$. We will look for the T-periodic solutions (u, v) of the system (HS) with the following boundary conditions

$$u(t, \partial\Omega) = v(t, \partial\Omega) = 0. \tag{1.4}$$

In order to receive the existence and multiplicity of T-periodic solutions of the asymptotically linear system (HS), we develop the theory of relative Morse index $(\mu_L(M), \upsilon_L(M))$ for the linearized system of (HS) by the relative Fredholm index theory, and display the relationship with Morse index of saddle point reduction and the Ekeland type index introduced by Dong in [17], where the concept of spectral flow will be used. To the best of the author's knowledge, the Morse index theory is the first time to be used to study the existence and multiplicity of periodic solutions for infinite dimensional Hamiltonian system.

Setting

$$J = \begin{pmatrix} 0 & -I_m \\ I_m & 0 \end{pmatrix}, \quad N = \begin{pmatrix} 0 & I_m \\ I_m & 0 \end{pmatrix}, \tag{1.5}$$

$$L := J\partial_t - N\Delta_x, \tag{1.6}$$