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A study on multivariate interpolation by increasingly flat kernel functions

Yeon Ju Lee [a], Charles A. Micchelli [b], Jungho Yoon [c,*]

[a] *Department of Mathematics, Korea University, Sejong, 339-700, South Korea*
[b] *Department of Mathematics and Statistics, State University of New York, The University at Albany, Albany, NY 12222, USA*
[c] *Department of Mathematics, Ewha Womans University, Seoul, 120-750, South Korea*

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ABSTRACT

In this paper, we improve upon some observations made in recent papers on the subject of increasingly flat interpolation. We shall establish that the corresponding Lagrange functions converge both for a finite set of functions (collocation matrix) and also for kernels (Fredholm matrix). In our analysis, we use a finite Maclaurin expansion of a multivariate function with remainder and some additional matrix theoretic facts.

1. Introduction

This paper is concerned with multivariate interpolation by a real-valued kernel function. We investigate the behavior of interpolation as the kernel function is increasingly flat. The motivation for this study is attributed to the interpolation by increasingly flat radial basis functions (RBFs). Many of commonly used RBFs ϕ (e.g., Gaussian) can be scaled in a way of being wider by a shape parameter (say, $\epsilon > 0$), by considering the function $\phi(\epsilon\cdot)$. It has been well-known that small values of $\epsilon > 0$ lead to accurate interpolants which are effective both for interpolation problems and for solving (elliptic and convective hyperbolic) partial differential equations [2,3,5,8–10]. In this case, the basis functions become flat so that the interpolation system becomes highly ill-conditioned, but the limit RBF interpolant is often well behaviored so that it converges to a finite number. Many recent studies have been devoted to finding the limiting behavior of the RBF interpolants as $\epsilon \to 0^+$. In [2], Driscoll and Fornberg observed numerically that the RBF interpolants converge towards multivariate polynomial interpolant as $\epsilon \to 0^+$, if the limit exists. Later, it has been

* Corresponding author.
E-mail addresses: leeyeonju08@gmail.com (Y.J. Lee), yoon@ewha.ac.kr (J. Yoon).

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verified that when the RBF ϕ is analytic, the RBF interpolants may converge to *polynomial interpolation* (under certain circumstances) to the data [11,14,16,17]. Limit polynomial may vary in a way of depending on the geometry of the center set and the given RBF. Moreover, when the unisolvency criterion does not meet (e.g, a tensor-product grid), the limit may *not* exist [2,11]. So, the convergence of flat interpolation needs yet to be fully investigated. Recently, interpolation by a finitely smooth RBF was studied in [13,18] where convergence as $\epsilon \to 0^+$ to *polyharmonic spline* interpolation was established.

The aim of this paper is to improve upon some observations made in recent papers on the subject of increasingly flat interpolation [2–18]. Almost all the previous works on this subject treat translation kernels formed by a radial function, except [12]. When translation kernels are considered it is assumed that the univariate function which generates the radial translation kernel is *analytic* in a neighborhood of the origin. For example, this is the case when the *Gaussian kernel* was considered in [17]. We shall *remove* both of these restrictions by identifying the *exact* smoothness needed, as well as by treating *arbitrary* kernels. To this end, we use a finite Maclaurin expansion of a multivariate function with remainder and some additional matrix theoretic facts. Then, we will see that when the kernel has a certain (finite) smoothness related to the number of the given centers, the interpolants converge to a polynomial interpolant as the kernel is increasingly flat. In fact, the work of [13] also deals with increasingly flat limit of interpolation by finitely smooth RBFs. It is worthwhile to note here that if the given kernel does not meet the smoothness condition, the interpolants by a finitely smooth RBF converge to an interpolant by a polyharmonic spline as $\epsilon \to 0^+$ [13].

2. Preliminaries

The following notation is used throughout this paper. We let $\mathbb{N}_n = \{1, 2, \ldots, n\}$ where n is a positive integer in $\mathbb{N}$. A lattice vector in $\mathbb{R}^d$ is a vector, each of whose coordinates is a nonnegative integer, and is typically denoted by $\alpha = (\alpha_j : j \in \mathbb{N}_d)$. That is, each coordinate α_j of α is in $\mathbb{Z}_+ := \mathbb{N} \cup \{0\}$. We denote the set of all lattice vectors in $\mathbb{R}^d$ as $\mathbb{Z}_+^d$. Corresponding to each $\alpha = (\alpha_j : j \in \mathbb{N}_d) \in \mathbb{Z}_+^d$ there is a monomial function m_α defined at $x = (x_j \in \mathbb{R} : j \in \mathbb{N}_d)$ as

$$m_\alpha(x) := x^\alpha := x_1^{\alpha_1} \cdots x_d^{\alpha_d}.$$

Likewise, we set $|\alpha|_1 := \sum_{j \in \mathbb{N}_d} \alpha_j$ and $\alpha! = \alpha_1! \cdots \alpha_d!$. A *homogeneous polynomial* of degree $k \in \mathbb{Z}_+$ on $\mathbb{R}^d$ is any linear combinations of the *monomials* in the set $\{m_\alpha : \alpha \in \mathbb{H}_k\}$ where we define

$$\mathbb{H}_k := \{\alpha : \alpha \in \mathbb{Z}_+^d, |\alpha|_1 = k\}.$$

We denote the set of all homogeneous polynomials of degree $k \in \mathbb{Z}_+$ on $\mathbb{R}^d$ by $\mathcal{H}_k$ and we recall that $\dim \mathcal{H}_k = \binom{k+d-1}{d-1}$. The space of all multivariate polynomials of *total degree* at most n is defined to be the linear span of the monomials in the set $\{m_\alpha : \alpha \in \mathbb{P}_n\}$ where we define

$$\mathbb{P}_n = \{\alpha : \alpha \in \mathbb{Z}_+^d, |\alpha|_1 \le n\}$$

and this set is denoted by $\mathcal{P}_n$. We also recall that the $\dim \mathcal{P}_n = \binom{n+d}{d}$. As we shall see, the *decomposition* of $\mathcal{P}_n$ into its *homogeneous parts* is *essential* in what follows. Also, central to our analysis is the *lexicographical* ordering of $\mathbb{Z}_+^d$. Specifically, we say $\alpha \prec \beta$, that is, α *comes before* β, provided either $|\alpha|_1 < |\beta|_1$, or $|\alpha|_1 = |\beta|_1$ and there is integer $m \in \mathbb{N}_d$ such that $\alpha_j = \beta_j$, $j \in \mathbb{N}_m$ and while $\alpha_{m+1} < \beta_{m+1}$ (assumed to be vacuously satisfied when $m = d$). Moreover, we use $\alpha \preceq \beta$ to mean that $\alpha \prec \beta$ or $\alpha = \beta$. Denoting by $\alpha^{[n]}$ the n-th lattice vectors in $\mathbb{Z}_+^d$, we use the symbol $\mathbb{B}_n$ for the set of all lattice vectors below or equal to $\alpha^{[n]}$ in the ordering "$\prec$", i.e.,

$$\mathbb{B}_n := \{\alpha : \alpha \in \mathbb{Z}_+^d, \alpha \preceq \alpha^{[n]}\}.$$