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Positive ground state solutions and multiple nontrivial solutions for coupled critical elliptic systems

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ABSTRACT

In this study, we consider the following coupled elliptic system with a Sobolev critical exponent:

$$\begin{cases} -\Delta u_1 + \lambda_1 u_1 = \nu_1 u_1^{p_1-1} + \mu_1 u_1^{2^*-1} + \beta u_1^{\frac{2^*}{2}-1} u_2^{\frac{2^*}{2}}, \ x \in \Omega, \\ -\Delta u_2 + \lambda_2 u_2 = \nu_2 u_2^{p_2-1} + \mu_2 u_2^{2^*-1} + \beta u_1^{\frac{2^*}{2}} u_2^{\frac{2^*}{2}-1}, \ x \in \Omega, \\ u_1, u_2 \geq 0 \text{ in } \Omega, \ u_1 = u_2 = 0 \text{ on } \partial\Omega, \end{cases} \qquad (\mathcal{P})$$

where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, $N \geq 5$, $2 < p_1, p_2 < 2^*$, $\lambda_j \in (-\lambda_1(\Omega), 0)$, $\mu_j > 0$ for $j = 1, 2$, and $\lambda_1(\Omega)$ is the first eigenvalue of $-\Delta$ with the Dirichlet boundary condition. We demonstrate the existence of a positive ground state solution for problem $(\mathcal{P})$ when the coupling parameter $\beta \geq -\sqrt{\mu_1\mu_2}$. Under some other conditions, we show the nonexistence of positive solutions for $(\mathcal{P})$ when $N \geq 3$. We also construct multiple nontrivial solutions and sign-changing solutions for the following system:

$$\begin{cases} -\Delta u_1 + \lambda_1 u_1 = \mu_1 u_1^3 + \beta u_2^2 u_1, \ x \in \Omega, \\ -\Delta u_2 + \lambda_2 u_2 = \mu_2 u_2^3 + \beta u_1^2 u_2, \ x \in \Omega, \\ u_1 = u_2 = 0 \text{ on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain and $N \leq 4$.

1. Introduction

In recent years, there have been extensive mathematical investigations of the following coupled elliptic system:

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$$\begin{cases} -\Delta u_1 + \lambda_1 u_1 = \mu_1 u_1^3 + \beta u_2^2 u_1, \ x \in \Omega, \\ -\Delta u_2 + \lambda_2 u_2 = \mu_2 u_2^3 + \beta u_1^2 u_2, \ x \in \Omega, \\ u_1 = u_2 = 0 \text{ on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain or $\Omega = \mathbb{R}^N$. System (1.1) arises when we consider the standing wave solutions to the following time-dependent Schrödinger system, which comprises two coupled Gross–Pitaevskii equations:

$$\begin{cases} -i\frac{\partial}{\partial t}\Phi_1 = \Delta\Phi_1 + \mu_1|\Phi_1|^2\Phi_1 + \beta|\Phi_2|^2\Phi_1, \ x \in \Omega, \ t > 0, \\ -i\frac{\partial}{\partial t}\Phi_2 = \Delta\Phi_2 + \mu_2|\Phi_2|^2\Phi_2 + \beta|\Phi_1|^2\Phi_2, \ x \in \Omega, \ t > 0, \\ \Phi_j = \Phi_j(x,t) \in \mathbb{C}, \ j = 1,2, \\ \Phi_j(x,t) = 0, \ x \in \partial\Omega, \ t > 0, \ j = 1,2, \end{cases} \tag{1.2}$$

which has many applications in physics and nonlinear optics (see [27]). Physically, the solution Φ_j denotes the jth component of the beam in Kerr-like photorefractive media (see [1]), μ_j represents self-focusing in the jth component, and the coupling constant β is the interaction between the two components of the beam. System (1.2) also arises in the Hartree–Fock theory for a double condensate, which is a binary mixture of Bose–Einstein condensates in two different hyperfine states $|1\rangle$ and $|2\rangle$ (see [18] and the references therein). In order to obtain solitary wave solutions of system (1.2), we set $\Phi_j(x,t) = e^{i\lambda_j t}u_j(x)$ for $j = 1,2$. Then, it is reduced to system (1.1). The existence of least energy and other finite energy solutions were studied by [2,6,7,13,15–17,20,25–27,30,31] and the references therein. The existence and multiplicity of positive and sign-changing solutions were studied by [2,5–7,10,11,13,20–24,26,29] and the references therein.

However, we note that most of the studies mentioned above considered the subcritical case, i.e., $N \le 3$. When $N = 4$, which is the critical case ($2^* := \frac{2N}{N-2} = 4$), the authors of [12] demonstrated the existence of a positive least energy solution for all negative β, positive small β, and positive large β. Later, in [14], they considered the following critical case for higher dimensional $N \ge 5$:

$$\begin{cases} -\Delta u_1 + \lambda_1 u_1 = \mu_1 u_1^{2^*-1} + \beta u_1^{\frac{2^*}{2}-1} u_2^{\frac{2^*}{2}}, \ x \in \Omega, \\ -\Delta u_2 + \lambda_2 u_2 = \mu_2 u_2^{2^*-1} + \beta u_1^{\frac{2^*}{2}} u_2^{\frac{2^*}{2}-1}, \ x \in \Omega, \\ u_1, u_2 \ge 0 \text{ in } \Omega, \ u_1 = u_2 = 0 \text{ on } \partial\Omega. \end{cases} \tag{1.3}$$

In [19], the authors also considered system (1.3) and showed that when $N \ge 3$, it has a positive solution for $\beta \to 0$ or $+\infty$. When $N = 4$, system (1.3) returns to (1.1). Interestingly, the authors of [14] obtained different results from the case when $N = 4$. Indeed, they showed that system (1.3) has a positive least energy solution for any $\beta \ne 0$. When $\beta = 0$, system (1.3) is reduced to the following critical exponent equation, which was first studied by Brezis and Nirenberg in [8]:

$$\begin{cases} -\Delta u_j + \lambda_j u_j = \mu_j u_j^{2^*-1}, \ x \in \Omega, \\ u_j \ge 0 \text{ in } \Omega, \ u_j = 0 \text{ on } \partial\Omega, \ j = 1,2. \end{cases} \tag{1.4}$$

In [8], they also included the following more general case:

$$\begin{cases} -\Delta u + \lambda_j u = \nu_j g_j(x,u) + \mu_j u^{2^*-1}, \ x \in \Omega, \\ u \ge 0 \text{ in } \Omega, u = 0 \text{ on } \ \partial\Omega. \end{cases} \tag{1.5}$$

By taking $g_j(x,u) = u^{p_j-1}$, where $2 < p_j < 2^*$ for $j = 1,2$, $N \ge 4$, $\nu_j, \mu_j > 0$, $\lambda_j \in (-\lambda_1(\Omega), 0)$, problem (1.5) has positive ground state solutions $v_1, v_2 \in C^2(\Omega) \cap C(\bar{\Omega})$ for $j = 1,2$ respectively, with the energy

$$0 < B_j := \frac{1}{2}\int_\Omega (|\nabla v_j|^2 + \lambda_j u_j^2) - \frac{1}{p_j}\int_\Omega \nu_j v_j^{p_j} - \frac{1}{2^*}\int_\Omega \mu_j v_j^{2^*} < \frac{1}{N}\mu_j^{-\frac{N-2}{2}} S^{\frac{N}{2}}, \tag{1.6}$$

where S is the sharp embedding constant from $D^{1,2}(\mathbb{R}^N)$ into $L^{2^*}(\mathbb{R}^N)$.