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Quasi-stationary distribution for the birth–death process with exit boundary

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ABSTRACT

We prove that there exists a unique quasi-stationary distribution for the minimal birth–death process with exit boundary. A spectral representation for the quasi-stationary distribution is also obtained.

1. Introduction and main results

Quasi-stationary distribution (QSD) for a Markov process describes the limiting behavior of an absorbing process when the process is conditioned to survive. The existence, uniqueness and other properties of quasi-stationary distributions for various Markov processes have been studied since 1940s. For survey on QSDs, refer to van Doorn and Pollett [9], Collet, Martínez and San Martín [4]. See Pollett [17] for an almost exhaustive bibliography for QSDs.

In this paper, we will study QSDs for the birth–death process with exit boundary. Consider a continuous-time birth–death process $X = (X_t, t \geqslant 0)$ taking values on nonnegative integers $\mathbb{Z}_+ = \{0, 1, \cdots\}$. Its Q-matrix is

$$Q = \begin{pmatrix} -(a_0+b_0) & b_0 & & & \\ a_1 & -(a_1+b_1) & b_1 & & \\ & a_2 & -(a_2+b_2) & b_2 & \\ & & \ddots & \ddots & \ddots \end{pmatrix}, \tag{1.1}$$

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where birth rates $\{b_i, i \geqslant 0\}$ and death rates $\{a_i, i \geqslant 1\}$ are positive, and $a_0 \geqslant 0$. When $a_0 = 0$ the process is irreducible; when $a_0 > 0$ the process can jump from state 0 to state -1 (say) and stay there.

Define

$$\mu_0 = 1, \quad \mu_i = \frac{b_0 \cdots b_{i-1}}{a_1 \cdots a_i}, \quad i \geqslant 1; \quad \nu_i = \frac{1}{\mu_i b_i}, \quad i \geqslant 0. \tag{1.2}$$

And let

$$A = \sum_{i=0}^{\infty} \nu_i, \quad B = \sum_{i=0}^{\infty} \mu_i, \quad R = \sum_{i=0}^{\infty} \nu_i \sum_{j=0}^{i} \mu_j, \quad S = \sum_{i=0}^{\infty} \nu_i \sum_{j=i+1}^{\infty} \mu_j. \tag{1.3}$$

When $a_0 > 0$ and $A = \infty$, the process is certainly absorbed at -1, van Doorn [6] identified all quasi-stationary distributions for the birth–death process. If ∞ is an entrance boundary (i.e. $R = \infty, S < \infty$), there is a unique QSD; if ∞ is a natural boundary (i.e. $R = S = \infty$), then either decay parameter $\lambda = 0$ and there is no QSD, or $\lambda > 0$ and there is a continuum of QSDs.

In [13], the authors gave a survey on the existence of QSDs for birth–death processes. They asked when ∞ is an exit boundary (i.e. $R < \infty, S = \infty$), do there exist QSDs for the minimal birth–death process? We give an affirmative answer. In this case, there exists a unique QSD for the minimal process.

When $R < \infty$ and $S = \infty$, the corresponding Q-process is not unique. Let $X = (X_t, t \geqslant 0)$ be the minimal Q-process with life time $T < \infty$, a.s. If $a_0 = 0$, T is the first leap time $\zeta := \lim_{n\to\infty} \zeta_n$, where ζ_n are the epochs of successive jumps: $\zeta_0 = 0$, $\zeta_n = \inf\{t : t > \zeta_{n-1}, X_t \neq X_{\zeta_{n-1}}\}$, $n \geqslant 1$. If $a_0 > 0$, $T = \zeta \wedge \tau$, where $\tau = \inf\{t > 0 : X_t = -1\}$. Let $P(t) = (p_{ij}(t), i, j \in \mathbb{Z}_+)$:

$$p_{ij}(t) = \mathbb{P}_i(X_t = j, t < T), \quad i, j \in \mathbb{Z}_+,\ t \geqslant 0, \tag{1.4}$$

be the Feller minimal transition function.

We will firstly formulate the following quasi-stationary distribution for the minimal process X. Let's denote the law of the process with initial distribution u by $\mathbb{P}_u$ (or $\mathbb{P}_i$ if $u = \delta_i$).

Definition 1.1. A proper probability distribution $u = (u_i, i \in \mathbb{Z}_+)$ is called a quasi-stationary distribution for the minimal process $(X_t, t \geqslant 0)$, if

$$\mathbb{P}_u(X_t = j | T > t) = u_j, \quad \forall j \in \mathbb{Z}_+,\ t \geqslant 0. \tag{1.5}$$

We will establish the connection between quasi-stationary distributions for the minimal process and the spectral properties of its generator Q. Since that the minimal Q-process is the only Q-process satisfying the forward equations, we can derive the following theorem from Theorem 4.1 in [16] (where the condition $q_{i0} > 0$ for some i is not needed).

Theorem 1.2. *Assume $a_0 \geqslant 0$, $R < \infty$ and $S = \infty$. Then $u = (u_i, i \in \mathbb{Z}_+)$ constitutes a quasi-stationary distribution for the minimal Q-process if and only if u solves $uQ = -\alpha u$ for some $\alpha > 0$, and satisfies $u_i > 0$, $\sum_{i\in\mathbb{Z}_+} u_i = 1$. Moreover, $\alpha = u_0 a_0 + \lim_{i\to\infty}(u_i b_i - u_{i+1} a_{i+1})$.*

Let λ be the decay parameter of the minimal birth–death process:

$$\lambda := -\lim_{t\to\infty} \frac{\log p_{ij}(t)}{t},\ i, j \geqslant 0. \tag{1.6}$$

It is well known that λ is independent of i, j. Let $g(\lambda) := (g_i(\lambda), i \in \mathbb{Z}_+)$ satisfy $Qg(\lambda) = -\lambda g(\lambda)$, then $g_i(\lambda) > 0$ for any $i \in \mathbb{Z}_+$. Denote $v_i = \mu_i g_i(\lambda)$, then $v = (v_i, i \in \mathbb{Z}_+)$ is λ-invariant measure for Q, that is $vQ = -\lambda v$. In [13, pages 192, 200], the authors asked that whether