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# On the existence of orthogonal polynomials for oscillatory weights on a bounded interval

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ABSTRACT

It is shown that the polynomials orthogonal on $(-1, 1)$ w.r.t. the oscillatory weight $e^{i\omega x}$ exist if $\omega$ is a transcendental number and $\tan\omega/\omega \in \mathbb{Q}$. Also, it is proved that such orthogonal polynomials exist for almost all $\omega > 0$, and the roots are all simple if $\omega > 0$ is either small enough or large enough.

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## 1. Introduction

We consider the problem of existence of orthogonal polynomials and Gaussian quadrature rules (in the standard form) for the following inner product:

$$(f, g)_\omega = \int_{-1}^{1} f(x)g(x)e^{i\omega x}\,dx, \tag{1}$$

with $\omega > 0$. A similar consideration on polynomials orthogonal on $(-1, 1)$ w.r.t. the weight function $x(1 - x^2)^{-1/2}e^{i\omega x}$ is the subject of the paper [3]. More precisely, we seek a monic polynomial $p_n^\omega$ of a given degree $n$ such that

$$\int_{-1}^{1} p_n^\omega(x)x^j e^{i\omega x}\,dx = 0, \qquad j = 0, 1, \dots, n - 1. \tag{2}$$

The following results on the existence of $p_n^\omega$ are due to [1]:

**Proposition 1.** *$p_1^\omega$ exists for any $\omega$ except when $\omega$ is a multiple of $\pi$.*

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**Proposition 2.** *$p_2^\omega$ exists for all $\omega$.*

**Conjecture 1.** *$p_n^\omega$ with $n$ even exists for all $\omega$.*

**Conjecture 2.** *$p_n^\omega$ with $n$ odd does not exist for some $\omega$.*

In this paper, we give a sufficient condition on $\omega$ for which $p_n^\omega$ exists for all $n$. According to Conjecture 1, this condition is not necessary. We show that $p_n^\omega$ exists for almost all $\omega > 0$. If the existence of $p_n^\omega$ is assumed, it is shown that all of its roots are simple when $\omega > 0$ is either small enough or large enough.

Throughout the paper, we frequently suppress the dependence of objects on $\omega$ for simplification in notations.

## 2. Orthogonal polynomials

A necessary and sufficient condition for existence of the orthogonal polynomial $p_n^\omega$ is that the Hankel determinant

$$\Delta_n = \begin{vmatrix} \mu_0 & \mu_1 & \cdots & \mu_{n-1} \\ \mu_1 & \mu_2 & \cdots & \mu_n \\ \vdots & \vdots & \cdots & \vdots \\ \mu_{n-1} & \mu_n & \cdots & \mu_{2n-2} \end{vmatrix} \tag{3}$$

does not vanish. The moment $\mu_k := \int_{-1}^{1} x^k \mathrm{e}^{\mathrm{i}\omega x}\,\mathrm{d}x$ is defined recursively (see [1]):

$$\mu_0 = \frac{2\sin\omega}{\omega}, \tag{4a}$$

$$\mu_k = \frac{1}{\mathrm{i}\omega}\left(\mathrm{e}^{\mathrm{i}\omega} - (-1)^k \mathrm{e}^{-\mathrm{i}\omega}\right) - \frac{k}{\mathrm{i}\omega}\mu_{k-1}, \quad k \ge 1. \tag{4b}$$

It is easy to show that

$$\mu_k = \frac{(-1)^k k!}{(\mathrm{i}\omega)^k} \sum_{\nu=0}^{k} \frac{(-\mathrm{i}\omega)^\nu s_\nu}{\nu!}, \tag{5}$$

where

$$s_\nu := \frac{1}{\mathrm{i}\omega}\left(\mathrm{e}^{\mathrm{i}\omega} - (-1)^\nu \mathrm{e}^{-\mathrm{i}\omega}\right) = \begin{cases} \dfrac{2\sin\omega}{\omega}, & \text{for } \nu \text{ even}, \\ \dfrac{2\cos\omega}{\mathrm{i}\omega}, & \text{for } \nu \text{ odd}. \end{cases}$$

Then we can expand (5) into

$$\mu_k = \frac{2(-1)^{k+1} k!}{(\mathrm{i}\omega)^k} \left( \cos\omega \sum_{\substack{\nu=1 \\ \nu \text{ odd}}}^{k} \frac{(-\mathrm{i}\omega)^{\nu-1}}{\nu!} - \frac{\sin\omega}{\omega}\left(1 + \sum_{\substack{\nu=2 \\ \nu \text{ even}}}^{k} \frac{(-\mathrm{i}\omega)^\nu}{\nu!}\right)\right). \tag{6}$$

Now consider the matrix corresponding to the Hankel determinant $\Delta_n$. If we take from the $r$th row the factor $\left(\frac{-1}{\mathrm{i}\omega}\right)^{r-1}$, and from the $s$th column the factor $\left(\frac{-1}{\mathrm{i}\omega}\right)^{s-1}$, then we arrive at a new Hankel determinant $\widetilde{\Delta}_n$ with the moments

$$\widetilde{\mu}_k := -2k! \left( \cos\omega \sum_{\substack{\nu=1 \\ \nu \text{ odd}}}^{k} \frac{(-\mathrm{i}\omega)^{\nu-1}}{\nu!} - \frac{\sin\omega}{\omega}\left(1 + \sum_{\substack{\nu=2 \\ \nu \text{ even}}}^{k} \frac{(-\mathrm{i}\omega)^\nu}{\nu!}\right)\right). \tag{7}$$

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