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Stochastic replicator dynamics subject to Markovian switching

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ABSTRACT

Population dynamics are often subject to random independent changes in the environment. For the two strategy stochastic replicator dynamic, we assume that stochastic changes in the environment replace the payoffs and variance. This is modeled by a continuous time Markov chain in a finite atom space. We establish conditions for this dynamic to have an analogous characterization of the long-run behavior to that of the deterministic dynamic. To create intuition, we first consider the case when the Markov chain has two states. A very natural extension to the general finite state space of the Markov chain will be given.

1. Introduction

Stochastic environments where independent external forces change the dynamic of the system are common in biological and economic settings [20,12,5,16,6,11,25,13,10,19,1]. An example to illustrate a complete and state-independent change in the dynamic is sickle-cell anemia [1,2]. Being a carrier for sickle-cell lowers an individual's fitness, however, during malaria outbreaks, since sickle-cell carriers have immunity, which increases their fitness. The random event of malaria outbreaks may be described as a continuous-time Markov chain that is independent of the population dynamic, yet changes the population dynamic.

Antibiotics affecting microbial populations is another example of a population subjected to an independent stochastic environment [4]. The authors discuss the affects of antibiotics to bacteria, such as *Escherichia coli* and *Salmonella enterica*, and what persistent environment is needed to support the type that is antibiotic resistant.

Although the motivation for our model is mostly biological, there is also a relationship with economics. A commonly used tool in economics is Markovian switching [12,5,16,6]. This includes modeling business cycles and GDP growth, electricity spot price models, and interest rates, all of which are agents working in a stochastic environment.

Kussell and Leibler [20] model the phenomenon of when bacteria change their phenotype to adjust to the stochastic environment. The authors assume there are n-phenotypes and linear growth where the fitness

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and switch to another phenotype is contingent on the current state of the environment. The stochastic environment is modeled by a k atom state continuous time Markov process, and is independent of the evolution of the population. In a current state, if a certain phenotype's fitness is comparatively small, this increases the probability of phenotype switching. The authors then derive the optimal long-term growth rate.

Markovian switching has also been applied to Lotka–Volterra and epidemiological population dynamics [11,25,28]. Gray et al. [11] assumed a deterministic susceptible–infected–susceptible model and changed parameters according to a continuous Markov chain. The authors discovered that the parameters coupled with the unique invariant measure of the Markov chain gave essentially new rates and found similar inequalities for either an endemic to occur or for the disease to become negligible. Takeuchi et al. [25] analyzed the switching between two deterministic Lotka–Volterra models and showed that this system is neither permanent nor dissipative. Zhu and Yin [28] consider a stochastic general Lotka–Volterra under Stratonovich-type perturbation, and show that this hybrid system has bounded growth rates, derive limits of certain long run averages, as well as derive conditions for almost-sure convergence within a two population system.

Fudenberg and L.A. Imhof [10] applied a simpler method where the event that switches the fitness of a population modeled by a Moran process was independent and identically distributed. The authors assumed a two state switched system, considered the mean of the fitness, and compared the switched fitnesses to derive their results.

Considering both a discrete Moran and a deterministic continuous time replicator dynamic, Harper et al. [13] applied a method similar to of Fudenberg and L.A. Imhof to determine whether the mean game of the switched system was either a strategy 1 dominant (prisoner's dilemma), strategy 2 dominant (prisoner's dilemma), coordination game, or mixed strategy dominant (hawk–dove). For the continuous time replicator dynamic, the authors determined this classification by comparing the ratios of the difference between the payoffs of the two underlying games and the ratio of whether the event will occur or not. These results are different than the ones derived in this paper.

We analyze a Markovian switched stochastic replicator dynamic with two strategies and determine conditions for this "new" game to be classified in one of the four games mentioned in the previous paragraph. The times between jumps to another state for the continuous time Markov chain are assumed to have an exponential distribution. Since the switched systems are stochastic, the classifications are similar to the ones given by Fudenberg and Harris [9], in that the inequality of a payoff of pure strategy against itself and the other strategy are perturbed by half the difference of the variances (perturbation from the white noise), and the *comparison* of the transition from a fixed state to the other states (perturbation from the Markov chain). Since the switching indirectly perturbs the dynamic, appropriately determined constants (that are not unique) are associated with a particular state. The difference between the constant of the fixed state and another state, multiplied by this transition rate, compares this transition. The sum of these terms encompasses the entire transition *comparison* for this state. For example, if the dynamic switches between two states where the transition rates are equal, then the addition/subtraction of an appropriately sized constant to the inequalities derived by Fudenberg and Harris [9] determine the proper inequalities for the dynamic.

To help create intuition, we first consider a Markov chain in a two atom state space, then extend the analogous results to the general finite atom state space. To illustrate the conditions for the long-run behavior, we give an example of cooperation in a stochastic environment where defection is punished in one environment, and not punished in the other. The efficacy of punishment is then explored.

2. Stochastic replicator dynamic

Consider a two-player symmetric game, where a_{ij} is the payoff to a player using pure strategy S_i against an opponent employing strategy S_j, and take $A = (a_{ij})$ as the payoff matrix. Within a population we