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Generalizations of generating functions for higher continuous hypergeometric orthogonal polynomials in the Askey scheme

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ABSTRACT

We use connection relations and series rearrangement to generalize generating functions for several higher continuous orthogonal polynomials in the Askey scheme, namely the Wilson, continuous dual Hahn, continuous Hahn, and Meixner–Pollaczek polynomials. We also determine corresponding definite integrals using the orthogonality relations for these polynomials.

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1. Introduction

In Cohl (2013) [3] (see (2.1) therein), we developed a series rearrangement technique which produced a generalization of the generating function for Gegenbauer polynomials. We have since demonstrated that this technique is valid for a larger class of orthogonal polynomials. For instance, in Cohl (2013) [2], we applied this same technique to Jacobi polynomials and in Cohl, MacKenzie and Volkmer (2013) [4], we extended this technique to many generating functions for Jacobi, Gegenbauer, Laguerre, and Wilson polynomials.

The series rearrangement technique starts by combining a connection relation with a generating function. This results in a series with multiple sums. The order of summations is then rearranged to produce a generalized generating function. This technique is especially productive when using connection relations with one free parameter. In this case, the connection relation is usually a product of Pochhammer symbols and

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the resulting generalized generating function has coefficients given in terms of generalized hypergeometric functions.

In this paper, we continue this procedure by generalizing generating functions for the remaining hypergeometric orthogonal polynomials in the Askey scheme [6, Chapter 9] with continuous orthogonality relations. We have also computed definite integrals corresponding to our generalized generating function expansions using continuous orthogonality relations. The orthogonal polynomials that we treat in this paper are the Wilson, continuous dual Hahn, continuous Hahn, and Meixner–Pollaczek polynomials. The generalized generating functions we produce through series rearrangement usually arise using connection relations with one free parameter. While connection relations with one free parameter are preferred for their simplicity, relations with more free parameters were considered when necessary.

Hypergeometric orthogonal polynomials with more than one free parameter, such as the Wilson polynomials, have connection relations with more than one free parameter. These connection relations are in general given by single or multiple summation expressions. For the Wilson polynomials, the connection relation with four free parameters is given as a double hypergeometric series. The fact that the four free parameter connection coefficient for Wilson polynomials is given by a double sum was known to Askey and Wilson as far back as in 1985 (see [5, p. 444]). When our series rearrangement technique is applied to cases with more than one free parameter, the resulting coefficients of the generalized generating function are rarely given in terms of a generalized hypergeometric series. The more general problem of generalized generating functions with more than one free parameter requires the theory of multiple hypergeometric series and is not treated in this paper. However, in certain cases when applying the series rearrangement technique to generating functions using connection relations with one free parameter, the generating function remains unchanged. In these cases, we have found that the introduction of a second free parameter can sometimes yield generalized generating functions whose coefficients are given in terms of generalized hypergeometric series (see for instance, Section 3 below and [2, Theorem 1]).

An interesting question regarding our generalizations is, "What is the origin of specific hypergeometric orthogonal polynomial generating functions?" There only exist two known non-equivalent generating functions for the Wilson polynomials, with the Wilson polynomials being at the top of the Askey scheme. Unlike the orthogonal polynomials in the Askey scheme which arise through a limiting procedure from the Wilson polynomials, most known generating functions for these polynomials do not arrive by this same limiting procedure from the two known non-equivalent generating functions for Wilson polynomials. All of the generating functions treated in this paper for the continuous Hahn, continuous dual Hahn and Meixner–Pollaczek polynomials, as well as most of those generating functions treated in our previous papers, do not arrive by this same limiting procedure from the Wilson polynomial generating functions. Therefore, the generalized generating functions for non-Wilson polynomials we present in this paper are interesting by themselves.

Here, we provide a brief introduction into the symbols and special functions used in this paper. We denote the real and complex numbers by $\mathbb{R}$ and $\mathbb{C}$, respectively. Similarly, the sets $\mathbb{N} = 1, 2, 3, \ldots$ and $\mathbb{Z} = 0, \pm 1, \pm 2, \ldots$ denote the natural numbers and the integers. We also use the notation $\mathbb{N}_0 = \{0, 1, 2, \ldots\} = \mathbb{N} \cup \{0\}$. If $a_1, a_2, a_3, \ldots \in \mathbb{C}$, and $i, j \in \mathbb{Z}$ such that $j < i$, then $\sum_{n=i}^{j} a_n = 0$, and $\prod_{n=i}^{j} a_n = 1$. Let $z \in \mathbb{C}$, $n \in \mathbb{N}_0$. Let $a_1, \ldots, a_p \in \mathbb{C}$, and $b_1, \ldots, b_q \in \mathbb{C} \setminus -\mathbb{N}_0$. The generalized hypergeometric function ${}_pF_q$ is defined as [7, Chapter 16]

$$
{}_pF_q\left(\begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix}; z\right) := \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{z^n}{n!}. \tag{1.1}
$$

If $p \le q$ then ${}_pF_q$ is defined for all $z \in \mathbb{C}$. If $p = q + 1$ then ${}_pF_q$ is defined in the unit disk $|z| < 1$, and can be continued analytically to $\mathbb{C} \setminus [1, \infty)$. The generalized hypergeometric function is used in the definitions