J. Math. Anal. Appl. 427 (2015) 399–427

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Perturbation analysis of embedded eigenvalues for water-waves

J.T. Kemppainen [a], S.A. Nazarov [b,c,1], K.M. Ruotsalainen [a,*]

[a] *University of Oulu, Mathematics Division, P.O. Box 4500, 90014 Oulu, Finland*
[b] *Mathematics and Mechanics Faculty, St. Petersburg State University – 198504, Universitetsky pr., 28, Stary Peterhof, Russia*
[c] *Saint-Petersburg State Polytechnical University, Laboratory Mechanics of New Nano-materials, Polytechnicheskaya ul., 29, St. Petersburg, 195251, Russia*

A R T I C L E   I N F O

Article history:
Received 10 October 2014
Available online 19 February 2015
Submitted by W.L. Wendland

Keywords:
Trapped modes
Water waves
Asymmetric waveguide
Perturbed domains
Asymptotic analysis

A B S T R A C T

The starting point of our study is the knowledge that certain surface piercing bodies support a trapped mode, i.e. an embedded eigenvalue in the continuous spectrum. In the framework of the two-dimensional theory of linear water waves, we investigate the question whether a trapped mode still exists after the small perturbation of the body contours. The perturbation of the obstacle is performed by a linear combination of appropriate profile functions. The coefficients of the profile functions and a perturbation parameter of the eigenvalue form a parameter space which controls the embedded eigenvalue as well as the geometry of the water domain. Based on the concept of enforced stability of embedded eigenvalues in the continuous spectrum, we will show that the trapped mode is preserved in the small perturbation, if the profile functions fulfil problem dependent orthogonalisation and normalisation conditions. The argumentation relies on a sufficient condition for the existence of a trapped mode and the notion of the augmented scattering matrix. With the help of asymptotic analysis, we will derive a fixed point equation in the parameter space to determine the appropriate profiles of perturbation. We study the solvability of this equation by the Banach contraction principle.

1. Introduction

1.1. Problem formulation

The motivation for our paper is the problem investigated in [16], where the interaction of water waves with obstacles was studied within the linear theory of water waves. It is related to the classic question whether the water wave problem admits a unique solution for all wave frequencies [10, Ch. 3]. In [16] an example of non-uniqueness is demonstrated. This has been achieved by constructing a trapped mode, i.e., a velocity

* Corresponding author.
E-mail addresses: Jukka.T.Kemppainen@oulu.fi (J.T. Kemppainen), srgnazarov@yahoo.co.uk (S.A. Nazarov), keijo.ruotsalainen@oulu.fi (K.M. Ruotsalainen).

[1] The second author was supported by the RFFI grant No. 15-01-02175.

http://dx.doi.org/10.1016/j.jmaa.2015.02.050

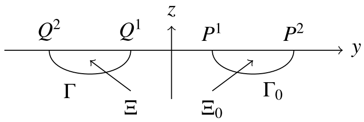

Fig. 1. The unperturbed domain.

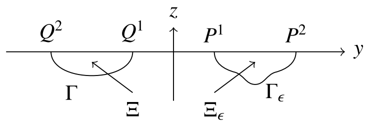

Fig. 2. The perturbed domain.

potential, which does not radiate any waves to infinity. The potential is formed by two wave sources with equal strength located symmetrically with respect to the origin. The two streamlines of the velocity potential can then be interpreted as the body contours of the surface piercing bodies. The constructed streamline patterns are symmetric. By the same method one can construct two surface piercing bodies, that are not symmetric, such that the water wave problem has a non-trivial solution with finite energy [10, p. 145]. We emphasise that the method in [16] requires rather specific shapes of the bodies.

Now the question we are raising is whether the trapped mode still exists after a small perturbation of the body contours. We start our investigation by assuming that a trapped mode is supported by the body contours Γ and Γ_0 which are smooth non-intersecting arcs in the lower half-plane

$$\mathbb{R}^2_- = \{(y,z):\ y \in \mathbb{R},\ z < 0\}$$

with the endpoints $Q^1 = (-b_1, 0), Q^2 = (-b_2, 0) \in \Gamma$ and $P^1 = (a_1, 0), P^2 = (a_2, 0) \in \Gamma_0$, respectively, where $a_2 > a_1 > 0$ and $b_2 > b_1 > 0$. We also need that the curves intersect $\partial\mathbb{R}^2_-$ under angles $\vartheta \in (0, \pi)$.

In what follows we assume that the arc Γ is fixed but Γ_0 has a small local perturbation. For that, we introduce in the neighbourhood $\mathcal{N}$ of Γ_0 local curvilinear coordinates (n, s), where n is the oriented distance to the curve Γ_0 and $s \in (0, l_0)$ is the arc length along Γ_0. Then the perturbed curve will be

$$\Gamma_\epsilon = \{(y,z) \in \mathcal{N}:\ n = \epsilon h(s),\ s \in (0, l_0)\} \tag{1}$$

depending on the small positive parameter $\epsilon > 0$ and on the profile function $h(s) \in C_0^\infty(0, l_0)$, which is an infinitely differentiable and compactly supported function and vanishes near the endpoints $s = 0$ and $s = l_0$. Therefore, the points P^1 and P^2 stay unperturbed.

By rescaling, the distance $|P^2 - P^1|$ can be set equal to one, which makes the Cartesian coordinates and the geometric parameters ϵ, l_0 and h dimensionless. Furthermore, by Ω_ϵ we denote the domain $\mathbb{R}^2_- \setminus (\overline{\Xi} \cup \overline{\Xi_\epsilon})$ (see Fig. 1 and Fig. 2), where Ξ and Ξ_ϵ are the domains bounded by the y-axis and the curves Γ and Γ_ϵ, respectively.

We will consider the standard two-dimensional linear water wave problem on the interaction of surface waves with the fixed obstacles Ξ and Ξ_ϵ, see [10]. As known, the velocity potential φ_ϵ satisfies the Laplace equation

$$-\Delta\varphi_\epsilon(y,z) = 0, \quad (y,z) \in \Omega_\epsilon. \tag{2}$$

On the wetted part of the obstacle surfaces it fulfils the Neumann boundary condition (no normal flow), i.e.

$$\partial_\nu\varphi_\epsilon(y,z) = 0, \quad (y,z) \in \Gamma \cup \Gamma_\epsilon, \tag{3}$$

and the kinematic boundary condition (the Steklov condition) on the free surface $\Sigma = \{(y,z):\ z = 0,\ y \notin [a_1, a_2] \cup [-b_2, -b_1]\}$,

$$\partial_z\varphi_\epsilon(y,0) = \lambda_\epsilon\varphi_\epsilon(y,0), \quad (y,0) \in \Sigma. \tag{4}$$

Here $\lambda_\epsilon = g^{-1}\omega_\epsilon^2$ is the spectral parameter, $g > 0$ is the acceleration due to gravity and $\omega_\epsilon > 0$ is the time frequency of propagating water waves. The outward normal derivative ∂_ν with respect to Ω_ϵ coincides with