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On co-polynomials on the real line

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ABSTRACT

In this paper, we study new algebraic and analytic aspects of orthogonal polynomials on the real line when finite modifications of the recurrence coefficients, the so-called co-polynomials on the real line, are considered. We investigate the behavior of their zeros, mainly interlacing and monotonicity properties. Furthermore, using a transfer matrix approach we obtain new structural relations, combining theoretical and computational advantages. Finally, a connection with the theory of orthogonal polynomials on the unit circle is pointed out.

1. Introduction

Let $d\mu$ be a non-trivial probability measure with an infinity support on some subset $A \subseteq \mathbb{R}$, such that

$$\int_A x^{2n} d\mu(x) < \infty, \quad n \geq 0.$$

The application of Gram–Schmidt's orthogonalization procedure to $\{x^n\}_{n\geq 0}$ yields a unique sequence of monic polynomials $\{P_n\}_{n\geq 0}$,

$$P_n(x) = x^n + (\text{lower degree terms}),$$

and a sequence, $\{\gamma_n\}_{n\geq 0}$, of positive real numbers such that

$$\int_A P_n P_m d\mu = \gamma_n \delta_{n,m}, \quad m \geq 0, \tag{1.1}$$

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where $\delta_{n,m}$ is the Kronecker delta. These polynomials are known in the literature as *orthogonal polynomials on the real line* (OPRL, in short), also known as *Chebyshev polynomials* before the book of Szegő [26] when the terminology was reserved for four special cases of trigonometric OPRL [26, Sec. 1.12].

It is very well known that the zeros of P_n, $\{x_{n,k}\}_{k=1}^{n}$, are real, simple and are located in the interior of the convex hull of the support A of the measure $d\mu$ and the zeros of P_n and P_{n+1} strictly interlace. The notation for zeros is

$$x_{n,n} < x_{n,n-1} < \cdots < x_{n,2} < x_{n,1}.$$

We suggest the reader to consult [2,7,13,20,21,26], where a complete presentation of the classical theory of OPRL can be found.

Associated with any sequence of OPRL there exist sequences $\{a_n\}_{n\geq 1}$ and $\{b_n\}_{n\geq 1}$ of positive real numbers and real numbers, respectively, such that

$$P_{n+1}(x) = (x - b_{n+1})P_n(x) - a_n P_{n-1}(x), \quad a_0 := 1, \quad n \geq 0, \tag{1.2}$$

with initial conditions $P_{-1} := 0$ and $P_0 := 1$. We set $P_n := 0$ for $n < 0$ and $a_n := b_n := 0$ for $n < 1$, then (1.2) holds for every $n \in \mathbb{Z}$. Set

$$\mathbf{P}_{n+1} := [\, P_{n+1}, P_n \,]^T, \quad \mathbf{A}_n := \begin{bmatrix} x - b_{n+1} & -a_n \\ 1 & 0 \end{bmatrix}.$$

Notice that from (1.2), we get

$$\mathbf{P}_{n+1} = \mathbf{A}_n \, \mathbf{P}_n, \quad \mathbf{P}_0 := [\, P_0, P_{-1} \,]^T,$$

as well as

$$\mathbf{P}_{n+1} = (\mathbf{A}_n \cdots \mathbf{A}_0)\, \mathbf{P}_0. \tag{1.3}$$

$\mathbf{A}_n$ is said to be the transfer matrix. This representation will be the central object in Section 3. The converse of the previous result is the so-called Favard's theorem or Spectral Theorem in the OPRL theory. In other words, given a sequence of polynomials, $\{P_n\}_{n\geq 0}$, generated by (1.2) with recurrence coefficients $\{a_n\}_{n\geq 1}$ positive real and $\{b_n\}_{n\geq 1}$ real numbers, then there exists a nontrivial probability measure $d\mu$ supported on the real line so that the orthogonality conditions (1.1) hold. Moreover, if $\{a_n\}_{n\geq 1}$ and $\{b_n\}_{n\geq 1}$ are bounded sequences, then $d\mu$ is unique. From now on, we will assume that the recurrence coefficients always satisfy the hypothesis of Favard's theorem.

The theory of OPRL has attracted an increasing interest from the pioneer works of Legendre, Gauss, Jacobi, Chebyshev, Christoffel, Stieltjes and Markov, among others. The construction of new sequences of OPRL by modifying the original sequence is a powerful tool, with many applications to theoretical and applied problems, such as asymptotic analysis, zero behavior, integrable systems, birth-and-death process, quadrature, and quantum mechanics, among others. In particular, the study of the properties of new sequences of OPRL with respect to finite modifications (by changing or shifting) of the recursion coefficients is a classical topic. For example, associated polynomials appear in Stieltjes' works [23,24] related to the convergence of certain continued fractions. Given the sequences $\{a_n\}_{n\geq 1}$ and $\{b_n\}_{n\geq 1}$, one defines for a fixed positive integer, k, the associated polynomials of order k, $\{P_n^{(k)}\}_{n\geq 0}$, by the recurrence relation

$$P_{n+1}^{(k)}(x) = (x - b_{n+k+1})P_n^{(k)}(x) - a_{n+k} P_{n-1}^{(k)}(x), \quad n \geq 0, \tag{1.4}$$