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Stochastic parabolic equations with nonlinear dynamical boundary conditions

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ABSTRACT

The existence and uniqueness of solutions to linear parabolic equations with nonlinear flux on the boundary driven by Gaussian boundary noise is studied. Both the case of heat equation with boundary conditions of Wentzell type and the case of white-noise boundary conditions are considered.

1. Introduction

Let $\mathcal{O} \subset \mathbb{R}^n$ be a bounded smooth domain with regular boundary $\Gamma = \partial\mathcal{O}$; throughout the whole paper we fix $T > 0$. We consider in $\mathcal{O}$ the classical heat equation

$$\dot{u}(t,x) - \Delta u(t,x) = 0 \qquad \text{in } (0,T) \times \mathcal{O} \tag{1}$$

with the initial condition

$$u(0,x) = u_0(x) \qquad \text{in } \mathcal{O} \tag{2}$$

that is used to describe a large variety of physical processes including heat conduction, fluid diffusion, population dynamics (e.g., see [9]) as well as for Kolmogorov's equation representation of Gaussian processes. When we get to boundary conditions, we consider that physical systems may be subject to a diffusion of heat energy also through the boundary, generated by the heat flux from the interior as well as from a white-noise type external source $b \cdot \dot{w}$ distributed on the boundary. This leads to stochastic dynamical

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boundary conditions, i.e., they involve a time derivative of the solution on the boundary and they have the form

$$\mathrm{d}u(t,\xi) + \left(\frac{\partial}{\partial\nu}u(t,\xi) + \gamma(u(t,\xi))\right)\mathrm{d}t = b(\xi)\cdot \mathrm{d}w(t) \qquad \text{on } (0,T)\times\Gamma \tag{3}$$

where $w(t) = \{w_i(t)\}_{i=1}^N$ is an N-dimensional Brownian motion on a stochastic basis $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{P})$ and $b = \{b_i\}_{i=1}^N \in L^\infty(\Gamma;\mathbb{R}^N)$. We denote by $\frac{\partial}{\partial\nu}$ the outward normal derivative to $\mathcal{O}$. In (3) the stochastic term is considered in the sense of Itô on $L^2(\Gamma)$.

As regards to the nonlinear function $\gamma : \mathbb{R}\to\mathbb{R}$ we assume here that

(H1) γ is continuous, monotonically non-decreasing such that $\gamma(0) = 0$ and

$$\gamma(r)\, r \geq a_1|r|^2 + a_2, \qquad \forall\, r\in\mathbb{R} \tag{4}$$

$$|\gamma(r)| \leq a_3|r| + a_4, \qquad \forall\, r\in\mathbb{R} \tag{5}$$

where $a_1 > 0$, $a_3 \geq 0$ and $a_2 \leq 0, a_4 \geq 0$.

Such kind of boundary conditions are already present in the mathematical literature, see for instance the seminal paper by Escher [14] for a nonlinear, deterministic model or, in a stochastic setting, Chueshov and Schmalfuß [8]. Although often dynamical boundary conditions are refereed to as *non-standard*, they have a natural derivation in the description of physical models with a dynamics on the boundary, such as heat transfer in a solid imbedded in a moving fluid [22, §7.4], surface gravity waves in oceanic models [11,12,18], as well as in fluid dynamics [21], phase separation phenomena [13], etc.

Problem ($\mathbf{P}_1$). We let ($\mathbf{P}_1$) be the stochastic problem described by the system of Eqs. (1)–(3).

Such a system is in fact the classical parabolic equation with Wentzell boundary conditions driven by a Gaussian noise on the boundary Γ,

$$\dot u(t,\xi) + \frac{\partial}{\partial\nu}u(t,\xi) + \gamma(u(t,\xi)) = b(\xi)\cdot\dot w(t) \qquad \text{on } (0,T)\times\Gamma. \tag{6}$$

(We refer to [6] for a related linear problem.) Also, the problem is related to the nonlinear stochastic diffusion equation studied in [7] where the diffusion is driven by a nonlinear operator div $\mathbf{a}(x,\nabla u)$ of Leray–Lions type (see also Section 6 below).

We shall prove in the next section the existence for the solution of problem ($\mathbf{P}_1$) via an operatorial approach which allows to rewrite system ($\mathbf{P}_1$) as a stochastic differential equation in the product space $H^1(\mathcal{O})\times L^2(\Gamma)$. A similar approach was recently developed for a class of deterministic parabolic equation with Wentzell boundary conditions in [4].

We consider also for $\varepsilon > 0$ and $0 < \alpha < \frac{1}{2}$ the problem ($\mathbf{P}_\varepsilon$) defined by equations

$$\begin{aligned}
&\frac{\partial u}{\partial t}(t,x) - \Delta u(t,x) = 0 \qquad \text{in } (0,T)\times\mathcal{O}\\
&\varepsilon\, du(t,x) + \varepsilon\,(-\Delta_\Gamma)^{1+\alpha}u\,dt + \frac{\partial}{\partial\nu}u(t,x)\,dt + \gamma(u(t,x))\,dt = b(\xi)\cdot\dot w(t), \quad \text{on } (0,T)\times\Gamma\\
&u(0,x) = u_0(x)
\end{aligned} \tag{7}$$

where $\Delta_\Gamma\colon L^2(\Gamma)\to L^2(\Gamma)$ is the Laplace–Beltrami operator on Γ, that is

$$\Delta_\Gamma u = \nabla_\Gamma\cdot(\nabla_\Gamma u), \quad \forall u\in D(\Lambda_\Gamma)$$