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Dynamic diffusion-type equations on discrete-space domains

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ABSTRACT

We consider a class of partial dynamic equations on discrete-space domains which includes, as a special case, the discrete-space versions of the diffusion (heat) equation. We focus on initial-value problems and study the existence and uniqueness of forward and backward solutions. Moreover, we discuss other topics such as sum and sign preservation, maximum and minimum principles, or symmetry of solutions.

1. Introduction

The classical diffusion (heat) equation $u_t = k\nabla^2 u$ describes the particle (or heat) distribution as a function of time. In this paper, we consider a class of diffusion-type equations with discrete space and arbitrary (continuous, discrete or mixed) time, namely

$$u^{\Delta_t}(x,t) = au(x+1,t) + bu(x,t) + cu(x-1,t), \quad x \in \mathbb{Z},\ t \in \mathbb{T}, \tag{1.1}$$

where $\mathbb{T}$ is a time scale (arbitrary closed subset of $\mathbb{R}$). The symbol u^{Δ_t} denotes the partial Δ-derivative with respect to t, which becomes the standard partial derivative u_t when $\mathbb{T} = \mathbb{R}$, and the forward partial difference $\Delta_t u$ when $\mathbb{T} = \mathbb{Z}$. Since the differences with respect to x never appear in this paper, we omit the lower index t in u^{Δ_t} and write u^{Δ} instead. We employ the time scale calculus to be able to study equations with continuous, discrete or mixed time domains in a unified way. Readers who are not familiar with the basic principles and notations of this recent mathematical tool are kindly asked to consult Stefan Hilger's original paper [10] or the survey monograph [7].

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Our motivation comes first and foremost from the absence of a systematic theory for discrete-space diffusion problems, although some more or less isolated facts about special cases of Eq. (1.1) can be found in the literature (see [3,11,12,26]). Moreover, the lack of suitable mathematical tools is in contrast with the use of (1.1) in applications:

- When $a = c$ and $b = -2a$, Eq. (1.1) represents a discretized version of the classical diffusion equation. Depending on the time scale, we can obtain the semidiscrete diffusion equation ($\mathbb{T} = \mathbb{R}$), or the purely discrete diffusion equation ($\mathbb{T} = \mathbb{Z}$).
- The case $a = 0$ and $0 < c = -b$ corresponds to the discrete-space transport equation.
- For $\mathbb{T} = \mathbb{Z}$, $a = c = 1/2$ and $b = -1$, Eq. (1.1) reduces to

$$u(x,t+1) = \frac{1}{2}u(x+1,t) + \frac{1}{2}u(x-1,t), \tag{1.2}$$

  which (together with the initial condition $u(0,0) = 1$ and $u(x,0) = 0$ for $x \neq 0$) describes the one-dimensional symmetric random walk on $\mathbb{Z}$ starting from the origin; the value $u(x,t)$ is the probability that the random walk visits point x at time t. More generally, consider a nonsymmetric random walk on $\mathbb{Z}$, where the probabilities of going left, remaining at the same position, or going right are $p, q, r \in [0,1]$, with $p + q + r = 1$. This random walk is described by Eq. (1.1), where $\mathbb{T} = \mathbb{Z}$, $a = p$, $b = q - 1$ and $c = r$. For $\mathbb{T} = \mathbb{R}$, we obtain a continuous-time Markov process which is similar to the well-known birth–death process (the difference is that in our situation, x can be positive as well as negative). Finally, for a general time scale $\mathbb{T}$, solutions of (1.1) can be regarded as heterogeneous stochastic processes.
- Applications of (1.1) go far beyond stochastic processes. For example, the semidiscrete diffusion equation appears in signal and image processing [15], while the discrete diffusion equation has been used to model mutations in biology [8].

From a theoretical point of view, our work could be perceived as a contribution to the study of partial dynamic equations (see, e.g., [2,11,12,16]). Alternatively, since we consider discrete space, our dynamic diffusion equations can be viewed as infinite systems of ordinary dynamic equations (see, e.g., [18]).

The paper is organized as follows. In Section 2, we provide some auxiliary results regarding the time scale exponential function. In Section 3, we study the existence and uniqueness of both forward and backward solutions. Section 4 deals with topics related to stochastic processes, such as space sum preservation, sign preservation, or maximum and minimum principles. In Section 5, we show that equations with symmetric right-hand sides possess symmetric solutions and characterize their maxima. Finally, in Section 6, we conclude the paper with a summarizing table and a set of open problems.

2. Preliminaries

Before we start our investigations of dynamic diffusion equations, it is necessary to present some auxiliary results concerning the time scale exponential function.

We need the time scale polynomials $h_k : \mathbb{T}^2 \to \mathbb{R}$, which are defined as follows:

$$h_0(t,s) = 1, \quad t,s \in \mathbb{T},$$

$$h_{k+1}(t,s) = \int_s^t h_k(\tau,s)\Delta\tau, \quad t,s \in \mathbb{T},\ k \in \mathbb{N}_0.$$