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Local minimizers in spaces of symmetric functions and applications

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A B S T R A C T

We study H^1 versus C^1 local minimizers for functionals defined on spaces of symmetric functions, namely functions that are invariant by the action of some subgroups of $\mathcal{O}(N)$. These functionals, in many cases, are associated with some elliptic partial differential equations that may have supercritical growth. So we also prove some results on classical regularity for symmetric weak solutions for a general class of semilinear elliptic equations with possibly supercritical growth. We then apply these results to prove the existence of a large number of classical positive symmetric solutions to some concave-convex elliptic equations of Hénon type.

1. Introduction

We study H^1 versus C^1 local minimizers for functionals defined in spaces of symmetric functions, namely functions that are invariant by the action of some subgroups of $\mathcal{O}(N)$, the group of $N \times N$ orthogonal matrices. The functionals considered in this paper may not be defined in the whole space $H_0^1(B)$, but on some proper subspaces of symmetric functions. Throughout in this paper B stands for the open unit ball centered at zero in $\mathbb{R}^N$, $N \geq 1$. In order to prove the equivalence in the C^1-topology and H^1-topology of local symmetric minimizers, it is essential to have the classical regularity for symmetric weak solutions of the Euler–Lagrange equations associated with these functionals. Problems with supercritical growth in the classical sense are involved and so classical regularity results, as in Brezis and Kato [7] based on the Moser's iteration technique [36], cannot be directly applied. By the same reason, the principle of symmetric criticality of Palais [39] does not apply. Hence we prove some regularity results, namely Theorems 2.2 and 2.5, which

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cover a large class of elliptic partial differential equations and extend and simplify the proofs of some results in [28, Sections 5.1 and 5.2].

We then apply these results to prove the existence of a large number of positive solutions to some classes of elliptic partial differential equations of concave-convex type. We prove the existence of at least three solutions and, if $N \geq 3$, up to $\left[\frac{N}{2}\right] + 2$ solutions, each of them exhibiting certain symmetry. In comparison with the pioneering work of Brezis and Nirenberg [9] and Ambrosetti et al. [3], our approach allows us to obtain the existence of more solutions and to treat problems that are critical or supercritical in the classical sense.

We consider elliptic equations of the type

$$-\Delta u = f(x,u) \quad \text{in} \quad B, \quad u = 0 \quad \text{on} \quad \partial B, \tag{1.1}$$

where f satisfies some suitable hypotheses regarding symmetry with respect to the first variable and growth that may even be supercritical in the classical sense. We also assume that f is Caratheodory, that is, for each $u \in \mathbb{R}$, $x \mapsto f(x,u)$ is measurable and, $u \mapsto f(x,u)$ is continuous for almost every $x \in B$.

As we will describe next, many interesting problems involving partial differential equations are invariant by the action of certain groups of symmetries and there are two major lines of research on this type of problems: the symmetry that solutions inherit from the problem, and the existence of solutions exhibiting the problem's symmetry.

On the first direction we mention the seminal work of Gidas et al. [23] in which, assuming quite sharp conditions on f, radial symmetry for any positive solution of (1.1) is proved. Bearing on this subject and related to the problems treated in this paper we also mention the results on symmetry breaking for least energy solutions of the Hénon equation [29], i.e. in case $f(x,u) = |x|^\alpha |u|^{p-1}u$ with $\alpha > 0$ and $p > 1$, proved in [43,11,13] and the results about the Schwarz foliated symmetry for least energy solutions proved in [42,38].

On the second direction, within which this paper contributes, the search of symmetric solutions naturally induces the study of spaces of symmetric functions. Here we mention the work of Strauss [45] on solitary waves; the work of Ni [37] on the Hénon equation; the work of Lions [34] about symmetry and compactness on Sobolev spaces; the work of de Figueiredo et al. [28] about embeddings of Sobolev spaces of symmetric functions in weighted L^p-spaces.

To state the results about H^1 versus C^1 local minimizers in spaces of symmetric functions, we introduce some notations: $F(x,u) := \int_0^u f(x,s)ds$, $\alpha \geq 0$ and

$$2^* = \begin{cases} 2N/(N-2) & \text{if} \quad N \geq 3, \\ \infty \quad \text{if} \quad N = 1,2, \end{cases} \qquad 2^*_\alpha = \begin{cases} 2(N+\alpha)/(N-2) & \text{if} \quad N \geq 3, \\ \infty \quad \text{if} \quad N = 1,2, \end{cases}$$

which are, in the case of $N \geq 3$, the critical exponents for the embeddings $H^1_0(B) \hookrightarrow L^p(B)$ and $H^1_{0,\text{rad}}(B) \hookrightarrow L^p(B, |x|^\alpha)$, respectively.

Theorem 1.1 *(H^1 versus C^1 local minimizers: space of radially symmetric functions). Assume the symmetry and growth conditions*

$$\begin{cases} f(x,u) = f(|x|,u), \quad \forall\, u \in \mathbb{R},\ \forall\, x \in B, \\ |f(x,u)| \leq C|x|^\alpha(1+|u|^q), \quad \forall\, x \in B, \quad \forall\, u \in \mathbb{R}, \quad C > 0 \textit{ is a constant}, \\ \alpha \geq 0,\ q = 2^*_\alpha - 1 \textit{ in case } N \geq 3 \textit{ and any } 1 < q \textit{ in case } N = 1,2, \end{cases} \tag{1.2}$$

and set

$$H^1_{0,\text{rad}}(B) = \{u \in H^1_0(B); u = u \circ O,\ \forall\, O \in \mathcal{O}(N)\} \quad \textit{and}$$
$$C^1_{0,\text{rad}}(\overline{B}) = \{u \in C^1(\overline{B}); u = u \circ O,\ \forall\, O \in \mathcal{O}(N) \textit{ and } u = 0 \textit{ on } \partial B\}.$$