J. Math. Anal. Appl. 429 (2015) 105–110

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Complex symmetry of invertible composition operators

Paul S. Bourdon [a], S. Waleed Noor [b,*]

[a] *Department of Mathematics, University of Virginia, Charlottesville, VA 22904-4137, USA*
[b] *Departamento de Matemática, ICMC-USP, São Carlos, SP 13566-590, Brazil*

ARTICLE INFO

Article history:
Received 3 February 2015
Available online 8 April 2015
Submitted by L. Fialkow

Keywords:
Complex symmetric operator
Composition operator
Normal operator
Linear fractional map
Denjoy–Wolff point
Elliptic automorphism

ABSTRACT

A bounded operator T on a separable Hilbert space $\mathcal{H}$ is said to be *complex symmetric* if there exists an orthonormal basis for $\mathcal{H}$ with respect to which T has a self-transpose matrix representation. In this paper, we study the complex symmetry of composition operators $C_\phi f = f \circ \phi$ induced on the Hardy space H^2 by holomorphic self-maps ϕ of the open unit disk $\mathbb{D}$. For *any* holomorphic self-map ϕ of $\mathbb{D}$, we establish that if C_ϕ is complex symmetric, then ϕ must fix a point in $\mathbb{D}$. Thus among the automorphisms of $\mathbb{D}$, only the elliptic ones may induce complex symmetric composition operators. For an elliptic automorphism ϕ, we prove that if ϕ is not a rotation or of order 3, then C_ϕ is complex symmetric if and only if

$$\phi(z) = \frac{\alpha - z}{1 - \bar{\alpha} z}$$

for some $\alpha \in \mathbb{D} \setminus \{0\}$.

1. Introduction

A bounded operator T on a separable Hilbert space $\mathcal{H}$ is *complex symmetric* if there exists an orthonormal basis for $\mathcal{H}$ with respect to which T has a self-transpose matrix representation. An equivalent definition also exists. A *conjugation* is a conjugate-linear operator $C : \mathcal{H} \to \mathcal{H}$ that satisfies the conditions

(a) C is *isometric*: $\langle Cf, Cg\rangle = \langle g, f\rangle \forall f, g \in \mathcal{H}$,

(b) C is *involutive*: $C^2 = I$.

We say that T is *C-symmetric* if $T = CT^*C$, and complex symmetric if there exists a conjugation C with respect to which T is C-symmetric.

* Corresponding author.
E-mail addresses: psb7p@virginia.edu (P.S. Bourdon), waleed@icmc.usp.br (S. Waleed Noor).

http://dx.doi.org/10.1016/j.jmaa.2015.04.008
0022-247X/

Complex symmetric operators on Hilbert spaces are natural generalizations of complex symmetric matrices, and their general study was initiated by Garcia, Putinar, and Wogen [7–10]. The class of complex symmetric operators includes a large number of concrete examples including all normal operators, binormal operators, Hankel operators, finite Toeplitz matrices, compressed shift operators, and the Volterra integral operator.

If X is a Banach space of holomorphic functions on the unit disk $\mathbb{D}$ and if ϕ is a holomorphic self-map of $\mathbb{D}$, the *composition operator* with *symbol* ϕ is defined by $C_\phi f = f \circ \phi$ for any $f \in X$. (The range of C_ϕ is contained in the space of holomorphic functions on $\mathbb{D}$, but need not be contained in X.) The study of composition operators consists in the comparison of the properties of the operator C_ϕ with those of symbol ϕ. For instance, if X is the Hardy space H^2, then every holomorphic self-map ϕ induces a bounded composition operator on H^2. The texts [4] and [11] contain comprehensive treatments of these operators.

The study of complex symmetry of composition operators was recently initiated by Garcia and Hammond [6]. There they pose the problem of characterizing all composition operators that are complex symmetric on H^2. The only non-constant symbols that currently are known to induce complex symmetric composition operators on H^2 are identified in [6]: $\phi(z) = \beta z$ for some β in the closure $\overline{\mathbb{D}}$ of $\mathbb{D}$ (i.e., precisely when C_ϕ is normal), and the order 2 elliptic automorphisms

$$\phi_\alpha(z) = \frac{\alpha - z}{1 - \bar{\alpha} z} \tag{1}$$

for $\alpha \in \mathbb{D}$. The complex symmetry of C_{ϕ_α} follows from $C^2_{\phi_\alpha} = C_{\phi_\alpha \circ \phi_\alpha} = I$ and a result of Garcia and Wogen [10, Thm. 2] which states that any bounded Hilbert space operator satisfying a polynomial equation of order 2 is complex symmetric. A conjugation J such that $C_{\phi_\alpha} = J C^*_{\phi_\alpha} J$ was found recently in [12].

Our main results are the following:

Proposition. *Let ϕ be an arbitrary holomorphic self-map of $\mathbb{D}$. If C_ϕ is complex symmetric, then ϕ must fix a point in $\mathbb{D}$.*

Theorem. *Let ϕ be an automorphism of $\mathbb{D}$ which is not a rotation or elliptic of order 3. Then C_ϕ is complex symmetric if and only if $\phi = \phi_\alpha$ for some $\alpha \in \mathbb{D} \setminus \{0\}$.*

The proposition above follows immediately from Proposition 2.1 of the next section, while the preceding theorem follows from Propositions 3.1 and 3.3 below, as well as [10, Thm. 2].

The plan of the paper is the following. In Section 2, we collect all the preliminary material we need. The main result here is Proposition 2.1, which says that if ϕ is any holomorphic self-map of $\mathbb{D}$ and C_ϕ is complex symmetric, then ϕ either has Denjoy–Wolff point in $\mathbb{D}$ or it is an elliptic automorphism. In Section 3, we study the complex symmetry of elliptic composition operators. Propositions 3.1 and 3.3 combined show that if ϕ is an elliptic automorphism of order ≥ 4 and is not a rotation, then C_ϕ is not complex symmetric. The order 3 elliptic case remains an *open problem.*

2. Preliminaries

2.1. The Hardy space H^2

Recall that a holomorphic function f on $\mathbb{D}$ belongs to the Hardy space H^p for some $0 < p < \infty$ if

$$||f||_p = \sup_{0 \leq r < 1} \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta \right)^{1/p} < \infty.$$