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# On the generalized Lebedev index transform

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A B S T R A C T

An essential generalization of the Lebedev index transform with the square of the Macdonald function is investigated. Namely, we consider a family of integral operators with the positive kernel $\left|K_{(i\tau+\alpha)/2}(x)\right|^2$, $\alpha \in \mathbb{R}$, $x > 0$, $\tau \in \mathbb{R}$, where $K_\mu(z)$ is the Macdonald function and $i$ is the imaginary unit. Mapping properties such as the boundedness, compactness, invertibility are investigated for these operators and their adjoints in weighted $L_p$ spaces. Inversion theorems are proved. Important particular cases are exhibited. As an interesting application, a solution of the initial value problem for the second order differential difference equation, involving the Laplacian, is obtained.



## 1. Introduction and preliminary results

Let $\alpha \in \mathbb{R}$. The main goal of this paper is to investigate mapping properties of a family of index transforms [7] and their adjoints in $L_p$, $p \geq 1$, involving the Macdonald function in the kernel, namely,

$$F_\alpha(\tau) = \int_0^\infty \left|K_{(i\tau+\alpha)/2}(x)\right|^2 f(x)dx, \quad \tau \in \mathbb{R}, \tag{1.1}$$

$$G_\alpha(x) = \int_{-\infty}^\infty \left|K_{(i\tau+\alpha)/2}(x)\right|^2 g(\tau)d\tau, \quad x \in \mathbb{R}_+, \tag{1.2}$$

where $i$ is the imaginary unit. The Macdonald function $K_\mu(z)$ [2], Vol. II, is the modified Bessel function of the second kind, which satisfies the differential equation

$$z^2\frac{d^2u}{dz^2} + z\frac{du}{dz} - (z^2+\mu^2)u = 0. \tag{1.3}$$

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It has the asymptotic behavior

$$K_\mu(z) = \left(\frac{\pi}{2z}\right)^{1/2} e^{-z}[1 + O(1/z)], \qquad z \to \infty, \tag{1.4}$$

and near the origin

$$z^\mu K_\mu(z) = 2^{\mu-1}\Gamma(\mu) + o(1),\ z \to 0, \tag{1.5}$$

$$K_0(z) = -\log z + O(1),\ z \to 0. \tag{1.6}$$

The Macdonald function can be represented by the integral

$$K_\mu(z) = \int_0^\infty e^{-z\cosh u}\cosh(\mu u)du,\ \mathrm{Re}\ z > 0,\ \mu \in \mathbb{C}. \tag{1.7}$$

Concerning the product of the Macdonald functions $K_{(\mu+\alpha)/2}(z)K_{(\mu-\alpha)/2}(z)$ the key formula, which will be used in the sequel is relation (2.16.5.4) in [5], Vol. II

$$K_{(\mu+\alpha)/2}(z)K_{(\mu-\alpha)/2}(z) = \int_0^\infty K_\mu\left(z\left(x + \frac{1}{x}\right)\right)x^{\alpha-1}dx, \quad \mathrm{Re}\ z > 0. \tag{1.8}$$

Letting in (1.1), (1.2) $\alpha = 0$, we come up with the operator of the Lebedev index transform and its adjoint, which is associated with the square of the Macdonald function [3]. We note that, indeed, an essential generalization of the Lebedev transform will be investigated, since it is impossible to reduce (1.1), (1.2) to the Lebedev operator via any substitution of parameters or functions.

## 2. Boundedness and compactness in Lebesgue's spaces

Let us introduce the following Lebesgue functional spaces

$$L^\alpha \equiv L_1\left(\mathbb{R}_+; K^2_{\alpha/2}(x)dx\right) := \left\{f : \int_0^\infty K^2_{\alpha/2}(x)|f(x)|dx < \infty\right\}, \tag{2.1}$$

which will be used in the sequel. In particular, as we will show below, it contains spaces $L_{\nu,p}(\mathbb{R}_+)$ for some $\nu \in \mathbb{R}$, $1 \le p \le \infty$ with the norms

$$||f||_{\nu,p} = \left(\int_0^\infty x^{\nu p-1}|f(x)|^p dx\right)^{1/p} < \infty, \tag{2.2}$$

$$||f||_{\nu,\infty} = \mathrm{ess\ sup}_{x\ge 0}|x^\nu f(x)| < \infty.$$

When $\nu = \frac{1}{p}$ we obtain the usual norm in $L_p$ denoted by $|| \ ||_p$.

**Lemma 1.** *Let $\alpha \in \mathbb{R}$, $\nu + \alpha < 1$, $1 \le p \le \infty$, $q = \frac{p}{p-1}$. Then the embedding holds*

$$L_{\nu,p}(\mathbb{R}_+) \subseteq L^\alpha \tag{2.3}$$

*and*