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# Nonlocal optimal design: A new perspective about the approximation of solutions in optimal design ☆

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ABSTRACT

It is well-known from the recent literature that nonlocal integral models are suitable to approximate integral functionals or partial differential equations. In the present work, a nonlocal optimal design model has been considered as approximation of the corresponding classical or local optimal control problem. The new model is driven by a nonlocal elliptic equation and the cost functional belongs to a broad class of nonlocal functional integrals. The purpose of this paper is to prove existence of optimal design for the new model. This work is complemented by showing that the limit of the nonlocal problem is the local one when the cost to minimize is the compliance functional (see [14]).



## 1. Introduction

It must be recalled that the Nonlocal Variational Formulation has served to derive interesting models in many fields of Applied Mathematics [6,7,11,12,17,23,32,35]. The operational method provided by the nonlocal framework has multiple advantages both from the practical and the theoretical point of view. The aim of this paper is to show the significance of the nonlocal formulation in order to face or pose interesting problems of optimization. In particular, this manuscript will address the application of this formulation to a type of Optimal Design problems. [19,20] are meaningful references where this approach has been carried out. Up to our knowledge these are the only works where specific problems of optimal control have been treated under a nonlocal perspective.

Our motivation is directly related to the difficulty that often takes place in the study of problems of optimal control. This difficulty usually lies in the lack of solutions or in the strict requirements expressed in

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the state equation (that mainly concerns with the assumed regularity for the solutions of the state equation). An interesting perspective about this type of trouble for optimal design problems can be consulted in [2,15,26] and [8,18,26–28] for some specific problems. To overcome some of those difficulties we propose a new model in which we replace the classical state equation and the cost functional by a model whose state equation and cost functional are written by means of nonlocal integrals. This seems to be a suitable way to approximate the solution (sometimes, relaxed solution) of the problem in the classical setting (see [19,20]). In any case, to undertake this strategy two issues are in order: existence and approximation results. Concerning the first issue we must point out that different frameworks and methodologies have been used to derive existence for a nonlocal integral equation. Among others, we may mention [1,9,22,24,36]. Concerning approximation by nonlocal functionals there is a wealth of literature on this subject, see for instance [4,5,10,13,24,31,36,37].

The present manuscript is inspired by the seminal paper [14] by Cea and Malanowski where an optimal design problem is considered. They dealt with the divergent-elliptic type equation

$$\begin{cases} -\operatorname{div}_x(h(x)\nabla u(x)) = f(x) \text{ in } \Omega \\ u \in H_0^1(\Omega) \end{cases} \tag{1.1}$$

and with the compliance as the cost functional to be minimized. It is an optimal design problem whose control or design variable, $h$, is the distribution of a conducting material within a fixed domain $\Omega$. The conductivity (or diffusion) is also $h$ (or a function of it) and the compliance, which is the energy dissipated in $\Omega$, is defined as

$$I(h,u) = \int_\Omega f(x)\,u(x)\,dx \tag{1.2}$$

where $u$ is the unique electrical potential (or temperature) that satisfies the divergent-elliptic equation (1.1).

Eq. (1.1) is a steady equation that has played an important role in Diffusion or in Conductivity Theory where, for some purposes, the compliance functional (1.2) has been considered as the objective to optimize (see [2,15,18]).

Here, in the present paper, as an introductory case, we shall study a nonlocal version of the problem (1.1)–(1.2). After that, and in a more general context, we shall address the question of existence of solution for optimal design problems described by means of a nonlocal elliptic state equation. The design function enters in the equation in a linear way and the cost belongs to a broad class of nonlocal functionals. The details are given below:

### *1.1. Formulation of the problem*

The present paper is concerned with the problem of design described as follows: the state equation, is

$$(P_\delta) \equiv \begin{cases} L_\delta(u) = f(x), \text{ in } \Omega \\ u = 0 \text{ in } \partial\Omega^{nl} \doteq \Omega_\delta - \Omega \end{cases} \tag{1.3}$$

where

$$L_\delta(u) = -2\int_{B(x,\delta)} H(x',x)\,\frac{k_\delta(|x'-x|)}{|x'-x|^2}\,(u(x')-u(x))\,dx',\ x\in\Omega$$

and