J. Math. Anal. Appl. 429 (2015) 311–325

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Rigorous numerical verification of uniqueness and smoothness in a surface growth model

Dirk Blömker, Christian Nolde *, James C. Robinson

ARTICLE INFO

Article history:
Received 2 December 2014
Available online 11 April 2015
Submitted by W. Layton

Keywords:
4th order nonlinear parabolic PDE
Global regularity
A-posteriori analysis
Surface growth equation

ABSTRACT

Based on numerical data and a-posteriori analysis we verify rigorously the uniqueness and smoothness of global solutions to a scalar surface growth model with striking similarities to the 3D Navier–Stokes equations, for certain initial data for which analytical approaches fail. The key point is the derivation of a scalar ODE controlling the norm of the solution, whose coefficients depend on the numerical data. Instead of solving this ODE explicitly, we explore three different numerical methods that provide rigorous upper bounds for its solution.

1. Introduction

We consider the following surface growth equation for the height $u(t,x) \in \mathbb{R}$ at time $t > 0$ over a point $x \in [0, 2\pi]$

$$u_t = -u_{xxxx} - ({u_x}^2)_{xx} \qquad x \in [0, 2\pi],\ t \in [0, T] \tag{1}$$

with periodic boundary conditions and subject to a moving frame, which yields the zero-average condition $\int_0^{2\pi} u(x,t)\,\mathrm{d}x = 0$.

This equation, usually with additional noise terms, was introduced as a phenomenological model for the growth of amorphous surfaces [21,18], and was also used to describe sputtering processes [6]; see [3] for a detailed list of references. Based on the papers [4,7,19] which develop the theory of 'numerical verification of regularity' for the 3D Navier–Stokes equations, our aim here is to establish and implement numerical algorithms to prove rigorously global existence and uniqueness of solutions of (1).

Despite being scalar the equation has surprising similarities to 3D Navier–Stokes equations [1–3]. It allows for a global energy estimate in L^2 and uniqueness of smooth local solutions for initial conditions in a critical Besov-type space that contains C^0 and $H^{1/2}$, see [3] (similar results for the 3D Navier–Stokes equations can

* Corresponding author.
E-mail addresses: dirk.bloemker@math.uni-augsburg.de (D. Blömker), christian.nolde@math.uni-augsburg.de (C. Nolde), J.C.Robinson@warwick.ac.uk (J.C. Robinson).

http://dx.doi.org/10.1016/j.jmaa.2015.04.025

be found in [9]). Here we focus on the one-dimensional model, since in this case more efficient numerical methods are available, and the calculations would be significantly slower in higher dimension. Moreover, for the two-dimensional case the situation of energy estimates seems even worse, as global existence could only be established in H^{-1} using the non-standard energy $\int_0^{2\pi} e^{u(x)}\,dx$, see [22] for details. Nevertheless, we believe that it should be possible to treat the 2D case using similar methods, but the analysis becomes more delicate since in two dimensions H^1 is the critical space (see [2,3]).

Rigorous methods for proving numerically the existence of solutions for PDEs are a recent and active research field. In addition to the approach taken here there are methods based on topological arguments like the Conley index, see [11,8,23], for example. For solutions of elliptic PDEs there are methods using Brouwer's fixed-point theorem, as discussed in the review article [17] and the references therein.

Our approach is based on [4] and similar to the method proposed in [14]. The key point is the derivation of a scalar ODE for the H^1-norm of the difference of an arbitrary approximation, that satisfies the boundary conditions, to the solution. The coefficients of this ODE depend only on the numerical data (or any other approximation used). As long as the solution of the ODE stays finite, one can rely on the continuation property of unique local solutions, and thus have a smooth unique solution up to a blowup time of the ODE. A similar approach using an integral equation based on the mild formulation was proposed in [12,13].

In order to establish a bound on the blow-up time for the ODE, one can either proceed analytically or numerically. We propose two analytical methods: one, based on the standard Gronwall lemma, enforces a 'small data' hypothesis and adds little to standard analytical existence proofs. The second is based on an explicit analytical upper bound to the ODE solution. A variant of this, a hybrid method in which one applies an analytical upper bound on a succession of small intervals of length $h > 0$ to the numerical solution and then restarts the argument, appears the most promising, and a formal calculation indicates that the upper bound from the third method in the limit of step-size to zero converges to the solution of the ODE.

In order to derive the ODE for the H^1-error, we use standard a-priori estimates. While the stability of the linear term $-u_{xxxx}$ means that these 'worst case' estimates are still sufficient, an interesting alternative approach in a slightly different context is proposed in [15,16], where the spectrum of the linearized operator (here $Lv = -v_{xxxx} + (v_x\varphi_x)_{xx}$, where φ is some given numerical data) is analyzed with a rigorous numerical method, which in the case of an unstable linear operator yields substantially better results, at the price of a significantly higher computational time. This will be the subject of future research.

The paper is organized as follows. In Section 2 we establish the a-priori estimates for the H^1-error between solutions and the numerical data, which in the end gives an ODE depending on the numerical data only. Section 3 provides the ODE estimates necessary for our three methods, while Section 4 states the main results. In the final Section 5, we compare our methods using numerical experiments.

2. A-priori analysis

In this section we establish upper bounds for the H^1-norm of the error

$$d(x,t) := u(x,t) - \varphi(x,t),$$

where u is a solution to our surface growth equation (1) and φ is any arbitrary, but sufficiently smooth approximation, that satisfies the boundary conditions. Since we know φ, if we can control the H^1 norm of d then we control the H^1 norm of u.

For the following estimates and results, we define the H^p-norm, $p \geq 1$, of a function u by

$$\|u\|_{H^p} := \|\partial_x^p u\|_{L^2},$$